Complex Analysis

II

المرحلة الرابعة

ست ايمان

Some methods of integration of complex functions

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Chapter (1)

Complex numbers

1- Sums and products

Def.- We define complex numbers as ordered pairs (x,y) of real numbers x and y.

Notations:-

① We denote complex number by $z, w, \ldots$

② We denote the set of all complex numbers by $\mathbb{C}$.

Def.- Let $z=(x,y)$ be a complex number

① x is called the real part z and denoted by $Re(z)$.

② y is called the imaginary part of z and denoted by $Im(z)$.

③ If $x=0$, then $z=(0,y)$ is called a pure imaginary number.

Ex:- Let $z=(-3,1)$

Then $Re(z)=-3$, $Im(z)=1$.

Remark:- $\forall x \in \mathbb{R}$, $x=(x,0) \in \mathbb{C}$ thus $\mathbb{R} \subset \mathbb{C}$

Def.- let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ be two complex numbers then

① $z_1=z_2$ if $x_1=x_2$ and $y_1=y_2$.

② $z_1+z_2=(x_1+x_2, y_1+y_2)$.

③ $z_1 z_2=(x_1x_2-y_1y_2, x_1y_2+x_2y_1)$.

Ex:- Show that $(2,-3)(-2,1) = (-1,8)$

Sol:- $(2,-3)(-2,1) = (-4-(-3), 2+6)$
$= (-1,8)$

Defi- We define the complex number $i = (0,1)$

Proposition:- let $i = (0,1)$ then

① $i^2 = -1$

② Every complex number $z = (x,y)$ can be written as
$z = x + iy$

Proof:- ① $i^2 = -1$

$i^2 = (0,1)(0,1) = (0-1, 0) = (-1,0) = -1$

② $z = (x,y) = x + iy$

let $z = (x,y)$, then

$x \in \mathbb{R}$, $x = (x,0)$

$y \in \mathbb{R}$, $y = (y,0)$

$\Rightarrow iy = (0,1)(y,0) = (0,y)$

$\Rightarrow x + iy = (x,0) + (0,y) = (x,y) = z$

Ex:- Find $z_1 + z_2$ and $z_1 z_2$ if $z_1 = 2-3i$ and $z_2 = -2+i$

Sol: $z_1 = 2-3i = (2,-3)$, $z_2 = -2+i = (-2,1)$

$z_1 + z_2 = 0 - 2i$

$z_1 z_2 = (2-3i)(-2+i)$

$= -4 + 2i + 6i - 3(i)^2$

$= -4 + 8i + 3$

$= -1 + 8i = (-1,8)$

2- Algebric properties:

Proposition:- Let z, z_1, z_2 and z_3 be complex numbers then

① $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$

② $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1(z_2 z_3) = (z_1 z_2) z_3$

③ $z_1(z_1 + z_3) = z_1 z_2 + z_1 z_3$

④ $0 + z = z + 0$ and $1z = z$

⑤ The additive inverse of $z = x + iy$ is $-z = -x - iy$

The multiplicative inverse of $z = x + iy \neq 0$ is

$$z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Proof: ① $z_1 + z_2 = z_2 + z_1$

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$= (x_2 + x_1) + i(y_2 + y_1)$ (since $\mathbb{R}$ is additive)

$= z_1 + z_2$

⑤ $z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

i.e $zz^{-1} = 1$

$$zz^{-1} = (x+iy)\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2}{x^2+y^2} - i\frac{xy}{x^2+y^2} + \frac{iyx}{x^2+y^2} + \frac{(i)^2 y^2}{x^2+y^2} = 1$$

Ex:- Find z^{-1} if $z = i$

Sol: $z^{-1} = i^{-1} = \frac{1}{i}\left(\frac{-i}{-i}\right) = -i$.

Ex: Find z^{-1} if $z = 2 - 3i$

Sol: $(2-3i)^{-1} = \frac{1}{2-3i}\left(\frac{2+3i}{2+3i}\right)$

$= \frac{2+3i}{4+6i-6i-9(i)^2}$

$= \frac{2+3i}{13} = \frac{2}{13} + i\frac{3}{13}$

Ex: Find $\frac{-1+2i}{-3+4i}$?

Sol: $\frac{-1+2i}{-3+4i} = \frac{-1+2i}{-3+4i}\left(\frac{-3-4i}{-3-4i}\right)$

$= \frac{3+4i-6i-8i^2}{9+12i-2i-16i^2}$

$= \frac{11-2i}{25}$

Proposition:- Let z_1 and z_2 be two complex numbers if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Proof: Assume $z_1 z_2 = 0$

Assume $z_1 \neq 0$.

$\Rightarrow z_1^{-1}$ exists

$\Rightarrow z_1^{-1}(z_1 z_2) = z_1^{-1}(0)$

$\Rightarrow (z_1^{-1} z_1) z_2 = 0$

$\Rightarrow z_2 = 0$.

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Ex:- Solve $z^2 + 3iz - 2 = 0$

Sol:- $z^2 + 3iz - 2 = 0$

$\Rightarrow (z + 2i)(z + i) = 0$

$z + 2i = 0$ or $z + i = 0$

$z = -2i$ or $z = -i$.

Ex:- Show that $z = -1 - 2i$ satisfies the equation $z^2 + 2z + 5 = 0$

Sol:- $z^2 + 2z + 5 = (-1 - 2i)^2 + 2(-1 - 2i) + 5$

$= 1 + 4i - 4 - 2 - 4i + 5$

$= 0$

Ex:- Show that $Re(z) = Im(iz)$

Sol:- let $z = x + iy$

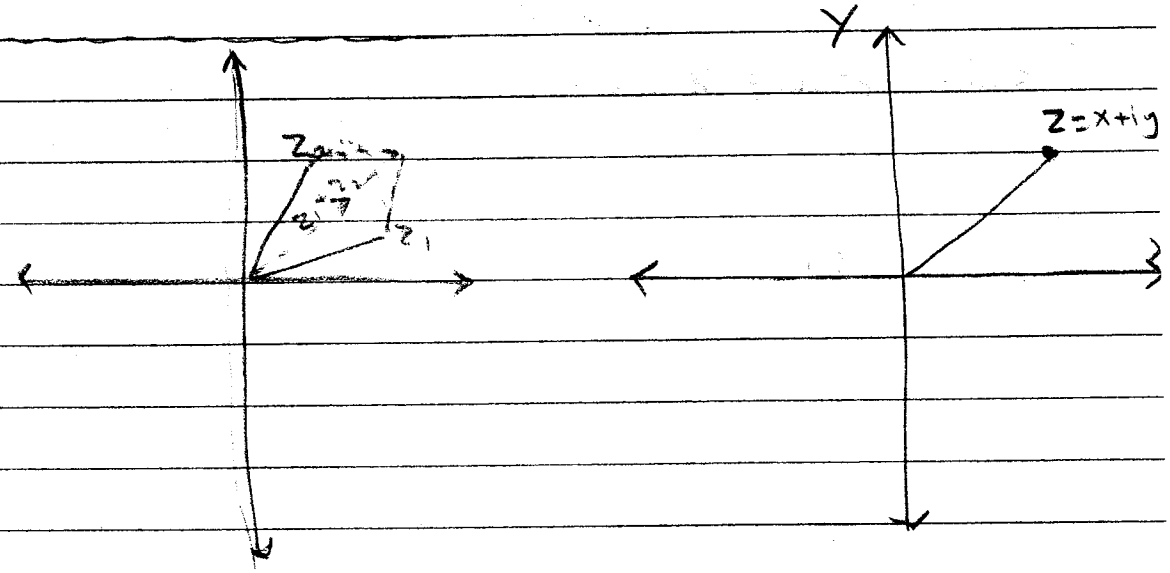
$\Rightarrow Re(z) = x$ ——— ①

$iz = i(x + iy) = ix - y = -y + ix$

$\Rightarrow Im(iz) = x$ ——— ②

From ① & ② $\Rightarrow Re(z) = Im(iz)$

3- Moduli and conjugate



Def:- The modulus (or absolute value) of a complex number $z = x+iy$ is

$$|z| = \sqrt{x^2+y^2}$$

Ex:- $z = 3-4i$, then

$$|z| = \sqrt{(3)^2+(-4)^2} = 5$$

Remarks:

① For any complex number z, $|z| \in \mathbb{R}$ and $|z| \geq 0$.

② $|z| = 0$ iff $z = 0$

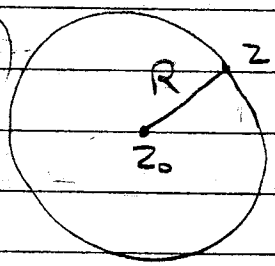
③ Geometrically, $|z|$ is the length of the vector representing z

④ $z_1 < z_2$ is meaning less unless both z_1 and z_2 are real numbers

⑤ $|z_1| < |z_2|$ means that the point z_1 is closer to $(0,0)$ than the point z_2.

⑥ $|z_1 - z_2|$ gives the distance between the points z_1 and z_2

② The equation of the circle with center z_0 and radius R is $|z - z_0| = R$.



Ex:- $|z - 2 + 3i| = 3$

$\Rightarrow |z - (2 - 3i)| = 3$

circle with center $(2, -3)$ and radius $R = 3$.

Proposition:- Let z be any complex number, then

① $Re(z) \leq |Re(z)| \leq |z|$

② $Im(z) \leq |Im(z)| \leq |z|$

Proof:- $|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$

$\Rightarrow |z|^2 = [Re(z)]^2 + [Im(z)]^2$

① $[Re(z)]^2 \leq |z|^2$

$\Rightarrow |Re(z)| \leq |z|$

But $Re(z) \leq |Re(z)| \leq |z|$.

Def:- The conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$

Ex:- if $z = 2 - 3i$, then $\bar{z} = 2 + 3i$

Proposition:- let z be a complex number, then

① $\bar{\bar{z}} = z$.

② $|\bar{z}| = |z|$

③ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \quad z_1, z_2 \in \mathbb{C}$

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④ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $z_1, z_2 \in \mathbb{C}$

⑤ $\left(\frac{z_1}{z_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$ $z_1, z_2 \in \mathbb{C}$

⑥ $Re(z) = \frac{z+\bar{z}}{2}$

⑦ $Im(z) = \frac{z-\bar{z}}{2i}$

⑧ $z\bar{z} = |z|^2$

⑨ $|z_1 z_2| = |z_1||z_2|$

⑩ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

proof: ④ let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

$\Rightarrow \overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$ ——— ①

$\bar{z}_1 \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$= x_1 x_2 - y_1 y_2 - i x_1 y_2 - i y_1 x_2$ ——— ②

⑤ $\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$

$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{z_1 \bar{z}_2}{|z_2|^2}\right)} = \frac{\bar{z}_1 \bar{z}_2}{|z_2|^2} = \frac{\bar{z}_1 z_2}{z_2 \bar{z}_2} = \frac{\bar{z}_1}{\bar{z}_2}$

⑥ $z = x + iy$, $\bar{z} = x - iy$

$z + \bar{z} = 2x$, $z - \bar{z} = 2iy$

⑧ $z\bar{z} = (x+iy)(x-iy)$

$= x^2 - ixy + iyx + y^2$

$= x^2 + y^2 = |z|^2$.

Ex:① Find $\frac{1}{3-4i} + \frac{3-2i}{6+8i}$

Ex:② Show that the hyperbola $y^2 - x^2 = 1$ can be written as $z^2 + \bar{z}^2 = -2$, where $z = x + iy$.

Sol, ① $\frac{1}{3-4i} + \frac{3-2i}{6+8i} = \left(\frac{1}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) + \left(\frac{3-2i}{6+8i}\right)\left(\frac{6-8i}{6-8i}\right)$

$= \frac{3+4i}{9+16} + \frac{18-24i-12i-16}{36+64}$

$= \frac{3+4i}{25} + \frac{2-36i}{100}$

$= \frac{7}{50} - \frac{10}{50}i.$

Sol. ② $x = Re(z) = \frac{z+\bar{z}}{2}$

$y = Im(z) = \frac{z-\bar{z}}{2i}$

$y^2 - x^2 = 1$

$\left(\frac{z-\bar{z}}{2i}\right)^2 - \left(\frac{z+\bar{z}}{2}\right)^2 = 1$

$\Rightarrow -\frac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2) - \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) = 1$

$\Rightarrow -\frac{1}{2}z^2 - \frac{1}{2}(\bar{z})^2 = 1$

$\Rightarrow z^2 + (\bar{z})^2 = -2.$

4. Triangle inequality :-

Proposition 1 :- (Triangle inequality)

Let z_1 and z_2 be complex numbers. Then

1) $|z_1 + z_2| \leq |z_1| + |z_2|$.

2) $|z_1 + z_2| \geq ||z_1| - |z_2|| \geq |z_1| - |z_2|$.

Example ①: Show that $|z_1 - z_2| \geq ||z_1| - |z_2||$.

Example ②: If z lies on the circle $|z| = 2$, then show that

$$3 \leq |z^2 - 1| \leq 5.$$

Proof: (Proposition) ① $|z_1 + z_2| \leq |z_1| + |z_2|$

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
&= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\
&= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} \\
&= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2| \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2
\end{aligned}$$

$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof ②: $|z_1 + z_2| \geq ||z_1| - |z_2||$

$$\Rightarrow -|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$$

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2|$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \quad \text{---} \quad (1)$$

in the same way we can show that

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

$$\Rightarrow |z_1| - |z_2| \geq -|z_1 + z_2| \quad \text{---} \quad (2)$$

Sol:- Ex(1) $|z_1 - z_2| \geq ||z_1| - |z_2||$

$$|z_1 - z_2| = |z_1 + (-z_2)| \geq ||z_1| - |z_2||$$

Sol:- Ex(2) $|z^2 - 1| \leq |z^2| + |-1| = |z|^2 + 1 = 4 + 1 = 5$

$$|z^2 - 1| \geq ||z^2| - 1|| = |4 - 1| = 3.$$

Ex(3): Using the inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$.
Show that if z lies on the circle $|z| = 3$, then
$|z^2 - 3z + 2| \geq 2$.

Sol:-

$$|z^2 - 3z + 2| = |(z-2)(z-1)| = |z-2||z-1|$$

$$|z-2| \geq ||z| - 2| = 1$$

$$|z-1| \geq ||z| - 1| = 2$$

$$\Rightarrow |z^2 - 3z + 2| \geq 2.$$

Ex: $z = 1 - i = (1, -1)$

$$r = |z| = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

$$\theta = \frac{-\pi}{4}, \quad z = r(\cos\theta + i\sin\theta)$$

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$1 - i = \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right].$$

5. Polar coordinates and Euler's formula:

Let r and θ be the point (x,y) that corresponds to the nonzero complex number $z = x+iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta)$$

Remark:-

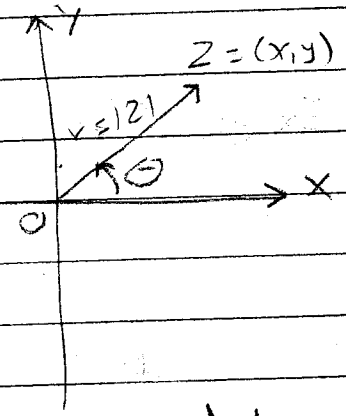
1. If $z = 0$, then θ is not defined and hence we can not write $z = 0$ in polar form.

2. In complex analysis we assume $r \geq 0$ and $r = |z| = \sqrt{x^2+y^2}$.

3. The real number θ represents the angle, measured in radians, that z makes with the positive real axis.

4. θ has an infinite number of values, both positive and negative, that differ by

$$2\pi.$$



5. The values of θ can be determined by specifying the quadrant containing $z = x+iy$ and by one of the following three equations.

$$\tan\theta = \frac{y}{x}, \quad \cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

Def:- Let z be a nonzero complex number.

1. Each value of θ such that
$$z = x+iy = r(\cos\theta + i\sin\theta)$$
is called an argument of z.
$\arg z = \{\theta : z = r(\cos\theta + i\sin\theta)\}$.

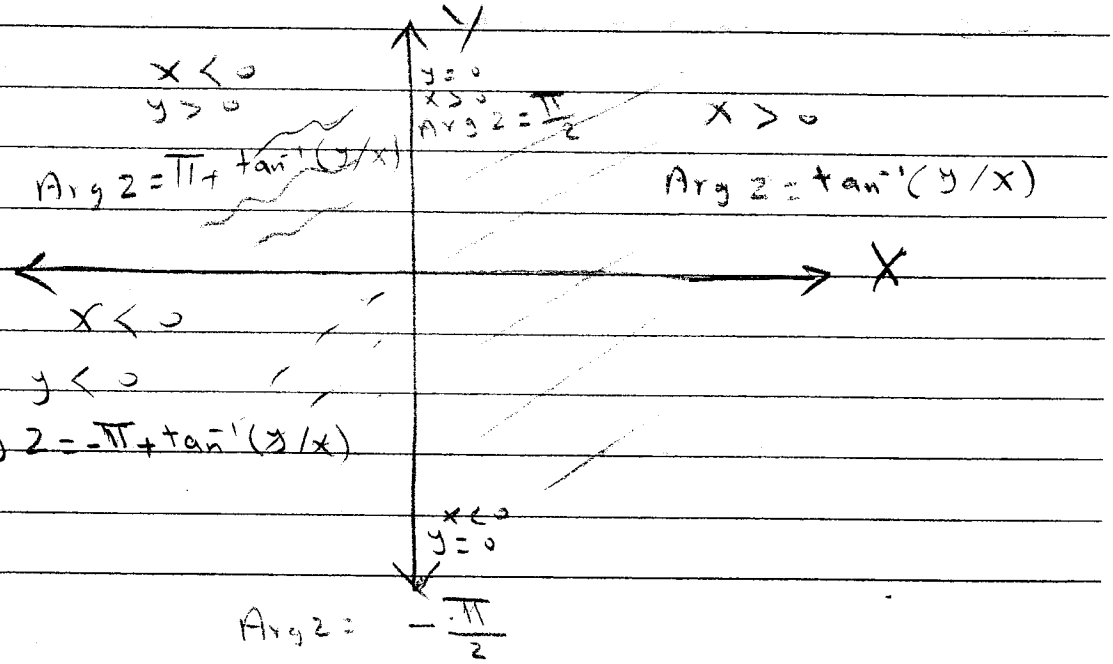
2. The set of all arguments of z is denoted by $\arg z$.

3. The principle value of $\arg z$ is the unique θ such that $-\pi < \theta \leq \pi$ and it is denoted by $\operatorname{Arg} z$.
$\operatorname{Arg} z \in (-\pi, \pi]$
$\arg z = \{\operatorname{Arg} z + 2n\pi , n \in \mathbb{Z}\}$.

Remark:-

(1) Note that $\arg z = \operatorname{Arg} z + 2n\pi$, $n \in \mathbb{Z}$.

(2) $$\operatorname{Arg} z = \begin{cases} \tan^{-1}(y/x) & x > 0 \\ \pi + \tan^{-1}(y/x) & x < 0,\ y \geq 0 \\ -\pi + \tan^{-1}(y/x) & x < 0,\ y < 0 \\ \pi/2 & x = 0,\ y > 0 \\ -\pi/2 & x = 0,\ y < 0 \end{cases}$$



Example: ① Find Arg (5), Arg (−5), Arg (2i), Arg(−3i).

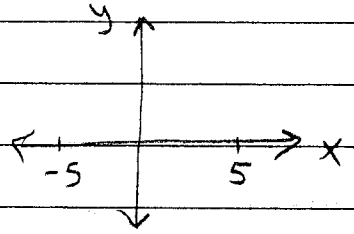
Example: ② Find arg (z) if $z = 1+i$.

Example: ③ Find the principle argument Arg z.

if $z = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$.

Sol:- ① $\text{Arg}(5) = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}(0) = 0$

$\text{Arg}(-5) = \pi + \tan^{-1}(0)$.



$\text{Arg}(2i) = \dfrac{\pi}{2}$.

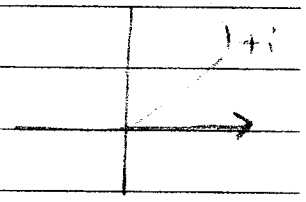
$\text{Arg}(-3i) = \dfrac{-\pi}{2}$.

Sol: ② $z = 1+i$

$\arg z = \text{Arg}\, z + 2n\pi, \quad n \in \mathbb{Z}$.

$\text{Arg}\, z = \tan^{-1}(y/x) = \tan^{-1}(1) = \dfrac{\pi}{4}$

$\arg z = \dfrac{\pi}{4} + 2n\pi, \quad n \in \mathbb{Z}$.



Sol: ③ العدد المركب الوحيد الذي لا يعطى بالصيغة القطبية هو الصفر

$z = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}} \cdot \dfrac{1+i\sqrt{3}}{1+i\sqrt{3}} = \dfrac{1-3+2\sqrt{3}\,i}{4} = \dfrac{-1+\sqrt{3}\,i}{2}$

$\text{Arg}(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\dfrac{\sqrt{3}/2}{-1/2}\right)$

$= \pi + \tan^{-1}(-\sqrt{3})$

$= \pi - \tan^{-1}\sqrt{3} = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$.

Euler's formula

For any real number θ, we will use the notation

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

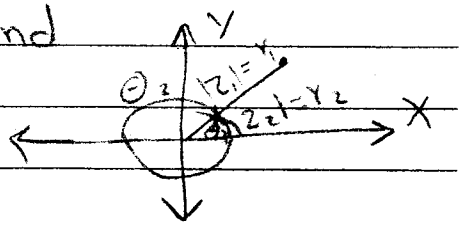
Thus the polar (or exponential) form of z can be written as:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\exp(i\theta).$$

Example ①: Write $z_1 = -1$ and $z_2 = 1+i$ in polar forms.

Example ②: Express $z = 3e^{\frac{\pi}{3}i}$ in the form $a+ib$.

Proposition ① Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi$, $n \in \mathbb{Z}$.



Sol: ① $z_1 = -1$, $z_2 = 1+i$

$$r_1 = |z_1| = |-1| = 1, \quad \theta_1 = \operatorname{Arg}(z_1) = \pi$$
$$\Rightarrow z_1 = -1 = e^{i\pi}.$$
$$z_2 \Rightarrow r_2 = |z_2| = \sqrt{2}$$
$$\theta_2 = \operatorname{Arg} z_2 = \frac{\pi}{4}$$
$$z_2 = 1+i = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

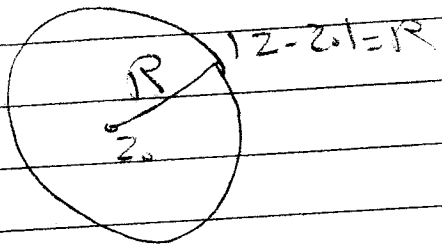
Sol ②: $z = 3e^{\frac{\pi}{3}i}$.

$$= 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$
$$= 3\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

Remark: $z = z_0 + Re^{i\theta}$, $0 \le \theta \le 2\pi$, is a parametric representation of the circle $|z - z_0| = R$.

$$z - z_0 = |z - z_0| e^{i\,\mathrm{Arg}(z - z_0)} = Re^{i\theta}$$

$$z = z_0 + Re^{i\theta}$$



Example ①: Show that $|e^{i\theta}| = 1$.

Sol.: $|e^{i\theta}| \overset{??}{=} 1$

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow |e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

6. Products and quotients in exponential form

proposition :- Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$,

2. $\frac{1}{z_1} = \frac{1}{r_1} e^{-i\theta_1}$,

proof: ① $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow z_1 z_2 = r_1[\cos\theta_1 + i\sin\theta_1] r_2(\cos\theta_2 + i\sin\theta_2)$.

$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$+ i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$.

proof ②: $\frac{1}{z_1} \overset{??}{=} \frac{1}{r_1} e^{-i\theta_1}$

$$\frac{1}{z_1} = \frac{1}{r_1 e^{i\theta_1}} \cdot \frac{e^{-i\theta_1}}{e^{-i\theta_1}} = \frac{e^{-i\theta_1}}{r_1 e^{i(\theta_1-\theta_1)}} = \frac{1}{r_1} e^{-i\theta_1}.$$

Ex: $z_1 = -1 - i$, $z_2 = \sqrt{3} - i$.

Sol: a) $r_1 = |z_1| = \sqrt{2}$

$\theta_1 = \text{Arg}\, z_1 = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$

$= -\pi + \tan^{-1}(1) = -\pi + \frac{\pi}{4} = -\frac{2\pi}{4}$

$z_1 = \sqrt{2} e^{-\frac{3\pi}{4}i}$

$r_2 = |z_2| = \sqrt{4} = 2$

$\theta_2 = \text{Arg}(z_2) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta_2 = -\frac{\pi}{6}$, $z_2 = 2e^{-\frac{\pi}{6}i}$.

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = 2\sqrt{2}\, e^{-i\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)} = 2\sqrt{2}\, e^{-\frac{7\pi}{12}i}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{\sqrt{2}}{2} e^{i\left(-\frac{3\pi}{4}+\frac{\pi}{6}\right)} = \frac{1}{\sqrt{2}} e^{-\frac{11\pi}{12}i}$

Proposition :- If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$, for any $n \in \mathbb{Z}$.

Remark :- (de Moivre's formula)
When $r=1$, the above formula becomes

$$(e^{i\theta})^n = e^{in\theta}.$$

This gives the useful de Moivre's formula.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example: Write $(\sqrt{3}+i)^5$ in the form $x+iy$.

proof: $z^n = r^n e^{in\theta}$, $n \in \mathbb{Z}$
by induction for $n \geq 0$

① $n=0$, $z^0 = r^0 e^{i(\theta \cdot 0)}$
$\Rightarrow 1 = 1$
If is true for $n=0$

② Assume it is true for k,
i.e $z^k = r^k e^{ik\theta}$
Then $z^{k+1} = z^k z$
$= (r^k e^{ik\theta})(re^{i\theta})$
$= r^{k+1} e^{i(k+1)\theta}$

therefore $z^n = r^n e^{in\theta}$ is truve
for all $n \geq 0$
For $n < 0$
$z^n = \left(\frac{1}{z}\right)^{-n} = \left(\left(\frac{1}{r}e^{-i\theta}\right)^{-n}\right.$

$n < 0 \Rightarrow -n > 0 \Rightarrow$

$z^n = \left(\frac{1}{r}\right)^{-n} e^{-i\theta(-n)}$

$= r^n e^{in\theta}$

$z^n = e^{in\theta} = (e^{i\theta})^n$

$\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Example ①:- Write $(\sqrt{3}+i)^5$ in the form $x+iy$.

Example:- (2) Use de Moivre's formula to derive the trignometric identities $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$.

Sol: (1) $(\sqrt{3}+i)^5 \overset{??}{=} x+iy$

$\sqrt{3}+i \overset{??}{=} re^{i\theta}$

$r = |\sqrt{3}+i| = 2$

$\theta = \text{Arg}(\sqrt{3}+i) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\Rightarrow \sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

$\Rightarrow (\sqrt{3}+i)^5 = \left(2e^{i\frac{\pi}{6}}\right)^5$

$= 32e^{i\frac{5\pi}{6}}$

$= 32\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

$= 32\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$

$= -16(\sqrt{3} - i)$.

Sol: (2) $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$.

$\Rightarrow \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos(2\theta) + i\sin(2\theta)$.

Proposition :- Let z_1 and z_2 be nonzero complex numbers. Then

(a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

(b) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

* $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Proof: (a) $\arg z_1 = \operatorname{Arg} z_1 + 2n_1\pi$

$\arg z_2 = \operatorname{Arg} z_2 + 2n_2\pi$

$\Rightarrow z_1 = r_1 e^{i \operatorname{Arg} z_1}$

$z_2 = r_2 e^{i \operatorname{Arg} z_2}$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)}$

But $z_1 z_2 = |z_1 z_2| e^{i \operatorname{Arg}(z_1 z_2)}$

$\Rightarrow r_1 r_2 e^{i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)} = r_1 r_2 e^{i \operatorname{Arg}(z_1 z_2)}$

$\Rightarrow \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi$.

$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1 z_2) + 2k\pi$.

$= \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi + 2k\pi$

$\arg(z_1 z_2) = \arg z_1 - 2n_1\pi + \arg z_2 - 2n_2\pi + 2k\pi + 2n\pi$.

$= \arg z_1 + \arg z_2 + 2(k + n - n_1 - n_2)\pi$.

Remark:

- The equations in the above proposition means that if particular values are assigned to any two of the three terms, then there is a value of the third term so that the equality holds
- The right and left hand sides of the equations in the above proposition coincide as sets.

Example: Let $z_1 = -1$ and let $z_2 = i$. show that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Sol:- $z_1 = -1$, $z_2 = i$, $z_1 z_2 = -i$

$\frac{z_1}{z_2} = \frac{-1}{i} = -i$

$\operatorname{Arg} z_1 = \operatorname{Arg}(-1) = \pi \Rightarrow \arg z_1 = \pi + 2n_1\pi$, $n_1 \in \mathbb{Z}$.

2)

$\operatorname{Arg} z_2 = \operatorname{Arg}(i) = \frac{\pi}{2} \Rightarrow \arg z_2 = \frac{\pi}{2} + 2n_2\pi$, $n_2 \in \mathbb{Z}$

$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \Rightarrow \arg(z_1 z_2) = -\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$

$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(i) = \frac{\pi}{2} \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} + 2m\pi$, $m \in \mathbb{Z}$.

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$-\frac{\pi}{2} + 2\pi = \frac{3\pi}{2} = \pi + \frac{\pi}{2}$ $(n_1 = n_2 = 0,\ n = 1)$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$??

$\frac{\pi}{2} = \pi - \frac{\pi}{2}$ $(n_1 = n_2 = m = 0)$.

Example:- Given that $z_1 z_2 \neq 0$, use the exponential forms of z_1 and z_2 to prove that $\operatorname{Im}(z_1 \bar{z}_2) = |z_1||z_2|$ if and only if $\arg z_1 - \arg z_2 = \frac{\pi}{2} + 2m\pi$, $m \in \mathbb{Z}$.

Sol: $\operatorname{Im}(z_1 \bar{z}_2) = |z_1||z_2|$

$\Leftrightarrow \arg z_1 - \arg z_2 = \frac{\pi}{2} + 2m\pi$, $m \in \mathbb{Z}$.

$z_1 = |z_1| e^{i \operatorname{Arg} z_1}$

$z_2 = |z_2| e^{i \operatorname{Arg} z_2}$

$\Rightarrow \bar{z}_2 = |z_2| e^{-i \operatorname{Arg} z_2}$

$\Rightarrow z_1 \bar{z}_2 = |z_1||z_2| e^{i(\operatorname{Arg} z_1 - \operatorname{Arg} z_2)}$

$\Rightarrow \operatorname{Im}(z_1 \bar{z}_2) = |z_1||z_2| \sin(\operatorname{Arg} z_1 - \operatorname{Arg} z_2)$

$\operatorname{Im}(z_1 \bar{z}_2) = |z_1||z_2|$

$\Leftrightarrow \sin(\operatorname{Arg} z_1 - \operatorname{Arg} z_2) = 1$

$\Leftrightarrow \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

$\Leftrightarrow \arg z_1 - 2n_1\pi - [\arg z_2 - 2n_2\pi] = \frac{\pi}{2} + 2n\pi$

$\Leftrightarrow \arg z_1 - \arg z_2 = \frac{\pi}{2} + 2\pi(\underbrace{n + n_1 - n_2}_{m})$

7. Roots of complex numbers

In this section we use that fact

$$z^n = |z|^n e^{i n \arg z}, \quad n \in \mathbb{Z}$$

to find then n-th roots of any nonzero complex number.

Def.- Let z_0 be nonzero complex number. An n-th root of z_0 is a nonzero complex number z such that $z^n = z_0$.

How can we find all the n-th roots of $z_0 \neq 0$?

Since $z_0 \neq 0$, it has a polar form

$$z_0 = r_0 e^{i\theta_0}$$

Similarly any n-th root z of z_0 is nonzero and hence it has a polar form

$$z = r e^{i\theta}.$$

Therefore any n-th root of z_0 has the form

$$z = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0 + 2k\pi)}{n}}, \quad k \in \mathbb{Z}$$

Note that $|z| = \sqrt[n]{r_0}$ for any n-th root z of z_0 this means that the roots lie on the circle.

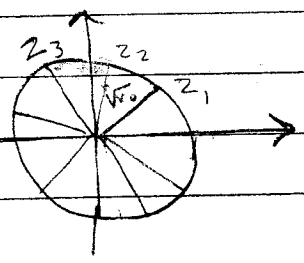
$$|z| = \sqrt[n]{r_0}.$$

Moreover they are equally spaced every $\frac{2\pi}{n}$ radians. Thus there are only n distinct roots.

These distinct roots can be obtained when

$$k = 0, 1, \ldots, n-1.$$

and they are denoted by z_k.



Therefore the n-th roots of $z_0 = r_0 e^{i\theta_0}$ are given by

$$c_k = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0 + 2k\pi)}{n}}, \quad k = 0, 1, \ldots, n-1.$$

Example ①:- Find the 3-rd roots of $z_0 = 1$.

Remark 1.

- For any complex number $z_0 \neq 0$, $z_0^{1/n}$ denotes the set of all n-th roots of z_0; that is

$$z_0^{1/n} = \{z : z^n = z_0\}.$$

Sol:① $z_0 = 1$, $(1)^{1/3} = ??$

$z_0 = 1$, $|z_0| = r_0 = 1$

$\theta_0 = \text{Arg}(z_0) = 0$

$z_0 = e^{i0} = 1$

The 3rd roots of $z_0 = 1$ are

$$c_k = \sqrt[3]{1}\, e^{\frac{i(0 + 2k\pi)}{3}}, \quad k = 0, 1, 2$$

$$c_k = e^{\frac{2\pi k i}{3}}$$

$c_0 = e^{0i} = 1$,

$c_1 = e^{\frac{2\pi}{3} i} = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$c_2 = e^{\frac{4\pi i}{3}} = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Remark 2.

- If r_0 is a positive real number, then $r_0^{1/n}$ denotes the set of all n-th roots of r_0,
$r_0^{1/n} = \{z : z^n = r_0\}$ but $\sqrt[n]{r_0}$ denote the unique positive real

n-th root of r_0.

• If $\theta_0 = \text{Arg } z_0$, then $c_0 = \sqrt[n]{r_0}\, e^{\frac{i\theta_0}{n}}$ is called the principal root of z_0.

The n-th roots of unity:

The n-th roots of 1 are given by

$$c_k = e^{\frac{2k\pi i}{n}}, \quad k = 0, 1, \dots, n-1.$$

Let $\omega_n = c_1 = e^{\frac{2\pi i}{n}}$ Then

$$c_2 = e^{\frac{4\pi i}{n}} = \omega_n^2, \dots, c_{n-1} = e^{\frac{2(n-1)\pi i}{n}} = \omega_n^{n-1}.$$

Therefore, the n-th roots of unity are

$$1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1},$$

where $\omega_n = e^{\frac{2\pi i}{n}}$.

Ex: $z^n = 1 \Rightarrow z^n = r_0 e^{i\theta}$

$z_0 = 1$

$c_k = \sqrt[n]{|z_0|}\, e^{i\left(\frac{\text{Arg } z_0 + 2k\pi}{n}\right)}$

$k = 0, 1, \dots, n-1$

$|z_0| = 1, \ \text{Arg } z_0 = 0, \ c_k = e^{\frac{i\,2k\pi}{n}}$

$c_0 = 1$

$c_1 = e^{\frac{2\pi i}{n}} = \omega_n, \quad c_{n-1} = \omega_n^{n-1}$

$c_2 = e^{\frac{4\pi i}{n}} = \omega_n^2$

$c_3 = e^{\frac{6\pi i}{n}} = \omega_n^3$

The n-th roots of z_0.

Let $\theta_0 = \text{Arg } z_0$. Then the n-th roots of $z_0 = r_0 e^{i\theta_0}$ are given by

$$c_k = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0 + 2k\pi)}{n}}, \quad k = 0, 1, \ldots, n-1.$$

They can be written as

$$c_k = \sqrt[n]{r_0}\, e^{\frac{i\theta_0}{n}} e^{\frac{2k\pi i}{n}} = c_0 w_n^k, \quad k = 0, 1, \ldots, n-1,$$

where $c_0 = \sqrt[n]{r_0}\, e^{\frac{i\theta_0}{n}}$ is the principle rout of z_0 and

$w_n = e^{\frac{2\pi i}{n}}$.

Example ①: Find the 2-nd roots of i.

Example ②: Find all values of $(2 + 2\sqrt{3}\, i)^{1/4}$.

Example ③: Find $(-4\sqrt{3} + 4i)^{1/3}$.

Sol: ① $i^{1/2} = ??$

$$c_k = c_0 w_2^k, \quad k = 0, 1.$$

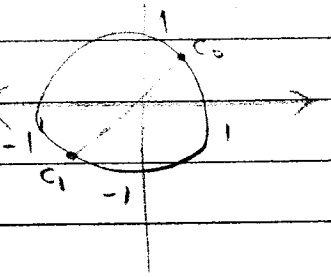
$$c_0 = \sqrt{|i|}\, e^{\frac{i\,\text{Arg}\, i}{2}} = e^{\frac{i\frac{\pi}{2}}{2}} = e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}$$

$$w_2 = e^{\frac{2\pi i}{2}}$$

$$w_2 = e^{\pi i} = -1$$

The routs of i are

$$c_0 = \frac{1+i}{\sqrt{2}}, \quad c_1 = c_0 w_2 = -c_0 = -\frac{(1+i)}{\sqrt{2}}.$$



Remark:- If n is a negative integer, then

$$z^{1/n} = \left(\frac{1}{z}\right)^{1/m},$$

where $m = -n$ is a positive integer.

Sol ②:- $z_0 = (2, 2\sqrt{3}\,i)$

The roots are

$$c_k = c_0 w_4^k \quad , k = 0, 1, 2, 3$$

$$c_0 = \sqrt[4]{|z_0|}\, e^{i\frac{\text{Arg } z_0}{4}}$$

$$= \sqrt[4]{4}\; e^{i\frac{\pi}{3}\cdot\frac{1}{4}}$$

$$c_0 = \sqrt{2}\; e^{i\frac{\pi}{12}}$$

$$w_4 = e^{\frac{2i\pi}{4}} = i$$

$\text{Arg } z_0 = \tan^{-1}(y/x)$
$= \tan^{-1}(\sqrt{3})$
$= \frac{\pi}{3}$

The roots are

$c_0,\ ic_0 = c_0 w_4,\ c_0 w_4^2 = -c_0,\ c_0 w_4^3 = -ic_0$

$c_1 \qquad c_2 \qquad c_3$

Sol ③:- $c_k = c_0 w_3^k, \ k = 0, 1, 2.$

$$c_0 = \sqrt[3]{r_0}\; e^{i\frac{\text{Arg } z_0}{3}}$$

$$r_0 = |z_0| = \sqrt{48+16} = 8$$

$$\text{Arg } z_0 = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{4}{-4\sqrt{3}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{5\pi}{6}$$

$$\Rightarrow c_0 = \sqrt[3]{8}\, e^{i\frac{5\pi}{18}}$$

$$c_0 = 2e^{i\frac{5\pi}{18}}$$

$$w_3 = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}i}{2}$$

Thus, the roots are.

$c_0, c_1 = c_0 w_3, c_2 = c_0 w_3^2$.

8. Regions in the complex plane.

In this section, we will study sets of points in the complex z-plane.

Def:- An ϵ-neighborhood of a point z_0 is the set

$$N_\epsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$$

consisting of all points lying inside the circle $|z - z_0| < \epsilon$

Def:- A deleted ϵ-neighborhood of a point z_0 is an ϵ-neighborhood of z_0 except z_0 itself. That is, it is the set

$$\{z \in \mathbb{C} \mid 0 < |z - z_0| < \epsilon\}$$

Def:- A point z_0 is said to be an interior point of a set S of complex numbers if there is an ϵ-neighborhood $N_\epsilon(z_0)$ of z_0 such that $N_\epsilon(z_0) \subseteq S$.

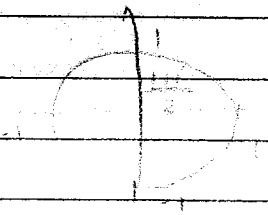
Def:- A point z_0 is an exterior point of a set S of complex numbers if there is an ϵ-neighborhood $N_\epsilon(z_0)$ of z_0 such that $N_\epsilon(z_0) \cap S = \phi$.

Def. A point z_0 is a boundary point of a set S if it is neither an interior nor an exterior point.

Def. The set of all boundary points of a set S is called the boundary of S.

Example: Let $S = \{z : |z| < 1\}$. Then the boundary of S is the set $S = \{z : |z| = 1\}$.

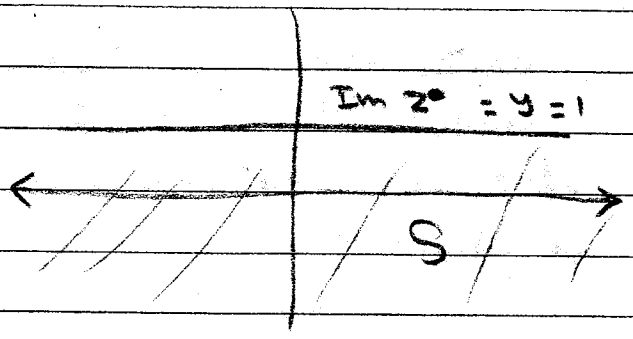
$z_1 = \frac{1+i}{2}$ is an interior point of S but $z_2 = 1+i$ is an exterior point of S.



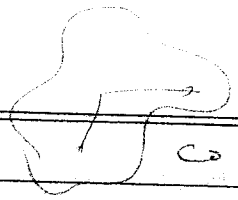
Def. Let S be a set of complex numbers.

1. S is open if it contains none of the boundary points.
2. S is closed if it contains all its boundary points.
3. The closure of S is the closed set consisting of all points in S together with the boundary of S.

Example: Let $S = \{z : \text{Im } z < 1\}$. Then S is open. The closure of S is the set $S = \{z : \text{Im } z \leq 1\}$.



Def:-



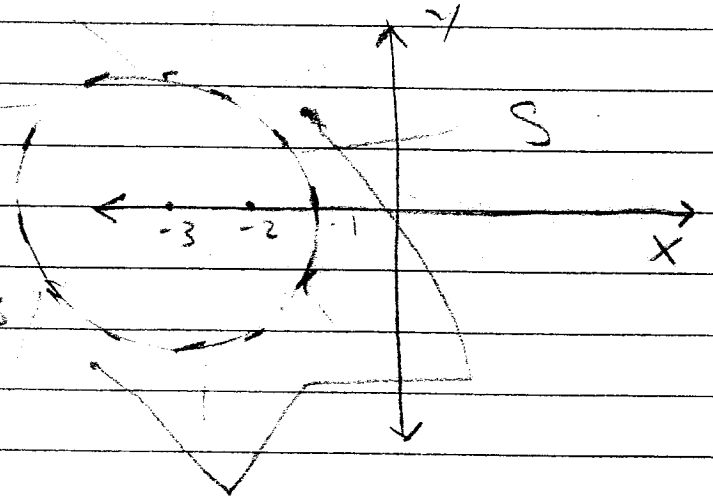
1. An open set S is said to be connected if each pair of points z_1 and z_2 in S can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.

2. A set S is called a domain if it is open and connected.

3. A region is a domain together with some, none, or all of its boundary.

Example: Sketch $S = \{z \in \mathbb{C} \mid |z+3| > 2\}$ and determin whether it is a domain or not.

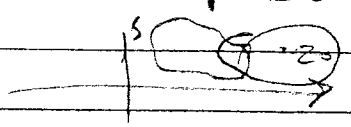
S is open set since all points of it are exterior point and connected since any points in S can joined a polygonal line



Def:- A set S is called to be bounded if there is a real number $R > 0$ such that $|z| < R$ for all $z \in S$.

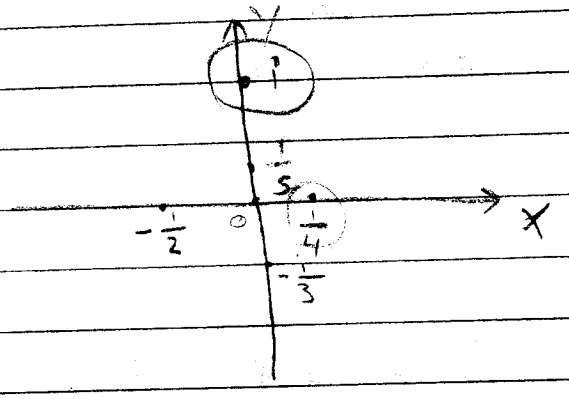
Def:- A point z_0 is called an accumulation point of a set S if each deleted neighborhood of z_0 contains at least one point of S.

accumulation



Example:- Let $S = \{ \frac{i^n}{n} \}$, $n = 1, 2, \ldots \}$.
Find the accumulation points of S.

$$S = \{ i, \frac{-1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{i}{5} \}$$



$$\left| \frac{i^n}{n} \right| = \frac{1}{n}$$

$$\left| \frac{i^n}{11} \right| = \frac{1}{-11} < \frac{1}{10}$$

∴ الصفر هي نقطة الترابط الوحيدة لأنه
كل النقاط تقترب إليها أما النقاط اليمينية
بمجرد أن صغر الجوار لن تنتهي غير النقطة
نفسها

Chapter 2

Analytic Functions

In this chapter, we consider functions of a complex variable and study continuity and differentiability of such functions. The main goal of this chapter is to introduce analytic functions.

Function of complex variable

Def: Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each $z \in S$ a single complex number w. In such case we write $w = f(z)$.

Remark 1.

1. If f is a function defined on S, then S is the domain of f.
2. If the domain is not given explicity, then we agree that its the largest possible set.

Example ① Let f be given by $f(z) = \frac{1}{z-1}$. Then f is a function of complex variable with domain $S = \mathbb{C} - \{1\}$.

Example ② Let f be given by $f(z) = \frac{1}{z^2+1}$. Then domain of f is $S = \mathbb{C} - \{i, -i\}$.

Real and imaginary parts of a function

Let $w = f(z)$, $z = x+iy$, be a function and let $w = u+iv$. Then we have

$$u+iv = f(x+iy).$$

Both of the real numbers u and v depends on x and y. Thus

$$f(z) = u(x,y) + i\,v(x,y).$$

$$z = re^{i\theta} \Rightarrow f(z) = u(r,\theta) + iv(r,\theta).$$

where u and v are real-valued functions of two real variables x and y.

Similarly, if $z = re^{i\theta}$, then

$$f(z) = u(r,\theta) + iv(r,\theta).$$

Example: Write $f(z) = z^2 + 2z$ in the forms $f(z) = u(x,y) + iv(x,y)$ and $f(z) = u(r,\theta) + iv(r,$

Sol: $z = x+iy$

$$\Rightarrow f(z) = z^2 + 2z$$
$$= (x+iy)^2 + 2(x+iy)$$
$$= x^2 - y^2 + 2xyi + 2x + 2yi$$
$$= x^2 - y^2 + 2x + (2xy + 2y)i$$
$$= u(x,y) + iv(x,y)$$

$z = re^{i\theta} \Rightarrow$

$$f(z) = z^2 + 2z$$
$$= r^2 e^{2i\theta} + 2re^{i\theta}$$
$$= r^2[\cos(2\theta) + i\sin(2\theta)] + 2r[\cos\theta + i\sin\theta]$$

=

$$= \underbrace{(r^2\cos(2\theta) + 2r\cos\theta)}_{u(r,\theta)} + i\underbrace{(r^2\sin(2\theta) + 2r\sin\theta}_{v(r,\theta)}$$

Remark 2. If $v(x,y) = 0$ for all z, then
$f(z) = u(x,y)$
is called a real-valued function of a complex variable.

Example. $f(z) = |z|^2 = x^2 + y^2$ is a real-valued function of a complex variable.

Polynomials and rational functions

(1) If n is a nonnegative integer, and if $a_0, a_1, \ldots, a_n$ are complex numbers such that $a_n \neq 0$, then the function

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

is a polynomial of degree n.

The domain of a polynomial function is the entire z-plane.

(2) If $P(z)$ and $Q(z)$ are polynomials, then

$$f(z) = \frac{P(z)}{Q(z)}$$

is a rational function.

The domain of a rational function $f(z) = \frac{P(z)}{Q(z)}$, is the set $\{z \in \mathbb{C} : Q(z) \neq 0\}$.

Example: ① $f(z) = iz^4 - (1+i)z^3 + 2z - 5 - 3i$, is polynomial of degree 4 whose domain is $\mathbb{C}$.

Example ②: $f(z) = \dfrac{2z^2 + 3iz + 4}{z^3 - (i+1)z^2 + iz}$

is rational function whose domain is

$\mathbb{C} \setminus \{0, 1, i\}$ $\begin{cases} z^3 - (i+1)z^2 + iz = 0 \Rightarrow z(z^2 - (1+i)z + i) = 0 \\ z^2 - (1+i)z + i = 0 \Rightarrow (z-i)(z-1) = 0 \end{cases}$

Example ③: Let $f(z) = x^2 + y^2 + 2i(x+y)$. Use

$x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$ to write $f(z)$ in terms

of z.

Sol: ② $z^3 - (i+1)z^2 + iz = 0$

$z(z^2 - (1+i)z + i) = 0$

$z^2 - (1+i)z + i = 0$

$(z-i)(z-1) = 0$

Sol: ③ $\left(\dfrac{z+\bar{z}}{2}\right)^2 + \left(\dfrac{z-\bar{z}}{2i}\right)^2 + 2i\left(\dfrac{z+\bar{z}}{2} + \dfrac{z-\bar{z}}{2i}\right)$

$= \dfrac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) - \dfrac{1}{4}(z^2 - 2z\bar{z} + \bar{z}^2)$

$+ i(z+\bar{z}) + z - \bar{z}$

$f(z) = z\bar{z} + z - \bar{z} + i(z+\bar{z})$.

$= |z|^2 + (1+i)z - (1-i)\bar{z}$.

Multi-valued functions

A multi-valued function is a rule that assigns more than one value to a point z in the domain of definition.

Remark 1.

1. Multivalued functions occur in the theory of functions of a complex variable just as they do in the case of real variables. They are naturally introduced as the inverse of single-valued functions.

2. When we study multi-valued functions, we choose only one of the possible values in a systematic manner, and we defined a continuous single-valued function. (ex. $f(z) = z^2$ multi-valu

Example 1. Let $z = e^{i\theta}$, $z \neq 0$. Then

$$z^{1/2} = \pm \sqrt{r}\, e^{i\theta/2}, \text{ where } -\pi < \theta = \operatorname{Arg} z \leq \pi.$$

If we choose only one of these two values, say

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2}, \quad -\pi < \theta < \pi,$$

then f is a single-valued function whose domain is the entire complex plane except for the ray $\theta = \pi$. (the negative real axis).

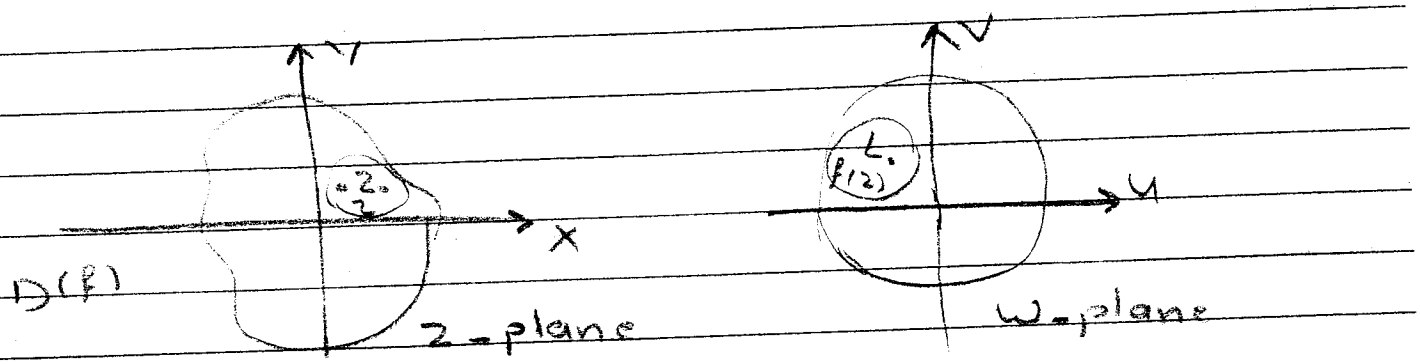
II. Limits

Def:- (Limit of a function): Let f be a function defined in some deleted neighborhood of a point $z_0 \in \mathbb{C}$. We say that the limit of $f(z)$ as z approaches z_0 is a number L, or

$$\lim_{z \to z_0} f(z) = L,$$

if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(z) - L| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$



Example ①:- Prove that $\lim_{z \to z_0} z = z_0$.

Sol:- $f(z) = z$, $L = z_0$.

Given $\epsilon > 0$, let $\delta = ??(\epsilon)$??

Then $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$

$|f(z) - L| = |z - z_0| < \delta = \epsilon$.

Example ②:- Show that $\lim_{z \to i} (3iz + 2) = -1$

$f(z) = 3iz + 2$, $z_0 = i$, $L = -1$

Let $\epsilon > 0$ be given and let $\delta \overset{??}{=} \frac{\epsilon}{3}$

Now $0 < |z - i| < \delta \Rightarrow$

$|f(z) - L| = |3iz + 2 + 1| = 3|iz + 1| = 3|i(z - i)| = 3|z - i||i|$

$< 3\delta = \epsilon$

Remark 2.

1. The definition of limit requires f to be defined in some deleted neighborhood of a point z_0. Such a deleted neighborhood always exists when z_0 is an interior point of the domain of f.

2. We can extend the definition of limit to the case in which z_0 is a boundary point of the domain of f. In this case we require

$$|f(z) - L| < \epsilon$$

only for z lies in both the domain of f and

$$0 < |z - z_0| < \delta.$$

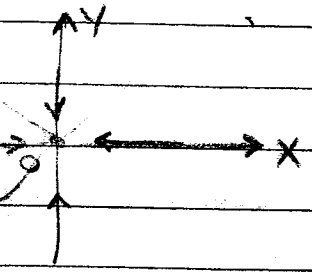
3. If $\lim_{z \to z_0} f(z) = L$ exists, then it is unique.

4. The symbol $z \to z_0$ means is allowed to approach z_0 is any arbitrary manner. Thus, if the limit approach different values from different directions, then the limit does not exist.

Example 1. Show that $\lim_{z \to 0} \frac{\bar{z}}{zz}$ does not exist.

Sol. If $z = x$, then

$$\lim_{z \to 0} \frac{\bar{z}}{zz} = \lim_{x \to 0} \frac{x}{x^2} = \frac{1}{2} \quad \text{---} ①$$



If $z = iy$, then

$$\lim_{z \to 0} \frac{\bar{z}}{zz} = \lim_{y \to 0} \frac{-iy}{iy^2} = -\frac{1}{2} \quad \text{---} ②$$

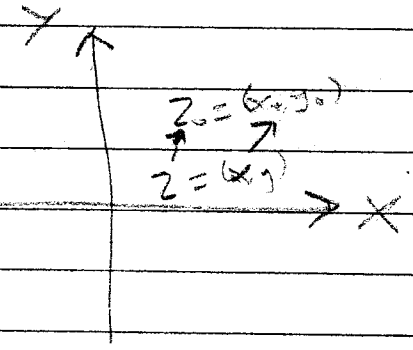
From ① & ② $\lim_{z \to 0} \frac{\bar{z}}{z^2}$ does not exist.

12. Theorems on limits

We can establish a connection between limits of $f(z)$ and limits of $u(x,y)$ and $v(x,y)$

Theorem: 1. Suppose that $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $L = A + iB$. Then

$$\lim_{z \to z_0} f(z) = L$$



if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = A$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = B.$$

proof: $\Rightarrow$) $\lim_{z \to z_0} f(z) = L \Rightarrow \lim_{(x,y) \to (x_0,y_0)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$

Given $\epsilon > 0$, let $\delta =$

$\lim_{z \to z_0} f(z) = L \Rightarrow \exists\, \delta_1 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - L| < \epsilon$$

Now, we want to prove $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$$|u(x,y) - A| < \delta$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$\Rightarrow 0 < |z - z_0| < \delta_1 = \delta$

and $|u(x,y) - A| = |Re(f(z) - L|$

$\leq |f(z) - L| < \epsilon$

$|v(x,y) - B| = |Im(f(z)) - L|$

$\leq |f(z) - L|$

proof:- $\Leftarrow$) H.W

Example 1. Find $\lim_{z \to 0} (z^2 + 2i)$.

sol:- $f(z) = z^2 + 2i = x^2 - y^2 + 2xyi + 2i$

$\Rightarrow u(x,y) = x^2 - y^2$, $v(x,y) = 2xy + 2$

$z \to 0 \Rightarrow (x,y) \to (0,0)$

$A = \lim_{(x,y)\to(0,0)} u(x,y) = \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0$

$B = \lim_{(x,y)\to(0,0)} v(x,y) = \lim_{(x,y)\to(0,0)} (2xy + 2) = 2$

$\therefore \lim_{z \to 0} (z^2 + 2i) = A + iB$

$= 2i$

Theorem 2. Assume that $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$.

Then

(a) $\lim_{z \to z_0} [f(z) \mp g(z)] = L \mp M$.

(b) $\lim_{z \to z_0} [f(z) g(z)] = LM$.

(c) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

proof:- Use the corresponding theorem for functions of two real variables.

Corollary 1.

(a) If $P(z)$ is a polynomail then

$$\lim_{z \to z_0} P(z) = P(z_0)$$

(b) If $P(z)$ and $Q(z)$ are polynomails and $Q(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}.$$

Example 1. $\lim_{z \to 2+i} (2z^3 + iz + 1)$.

Sol:- $\lim_{z \to 2+i} (2z^3 + iz + 1) = 2(2+i)^3 + i(2+i) + 1$

$= 2(8 + 12i - 6 - i) + 2i - 1 + 1$

$= 4 + 24i$

Proposition 1. If $\lim_{z \to z_0} f(z) = L$, then

$$\lim_{z \to z_0} |f(z)| = |L|.$$

proof:- Assume $\lim_{z \to z_0} f(z) = L$

Let $\epsilon > 0$ be given and let $\delta =$

$\lim_{z \to z_0} f(z) = L \Rightarrow \exists \delta_1 > 0$ such that

$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - L| < \epsilon.$

Assume $0 < |z - z_0| < \delta = \delta_1$
Then $||f(z) - L|| \leq |f(z) - L| < \epsilon$.

13. Limits involving the point at infinity

It is oftn useful to include the point at infinity denoted by ∞, with the complex plane.

Def. (Extended complex plane)
The extended complex plane is the complex plane together with the point at infinity (∞).

Def. (Infinite limits)

(a) We say that $\lim_{z \to z_0} f(z) = \infty$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\epsilon},$$

(b) We say that $\lim_{z \to \infty} f(z) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z| > \frac{1}{\delta} \Rightarrow |f(z) - L| < \epsilon.$$

(c) We say that $\lim_{z \to \infty} f(z) = \infty$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z| > \frac{1}{\delta} \Rightarrow |f(z)| > \frac{1}{\epsilon}.$$

Proposition 2.

(a) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

(b) $\lim_{z \to \infty} f(z) = L$ if and only if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = L$.

(c) $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$

proof: (a) $\lim_{z \to z_0} f(z) = \infty \Rightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0$

Assume $\lim_{z \to z_0} f(z) = \infty$

Let $\epsilon > 0$ be given and let $\delta =$

$\lim_{z \to z_0} f(z) = 0 \Rightarrow \exists \delta_1 > 0$ such that

$0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\epsilon}$

Assume $0 < |z - z_0| < \delta = \delta_1$. Then

$\left| \frac{1}{f(z)} - 0 \right| = \frac{1}{|f(z)|} < \epsilon$ (since $|f(z)| > \frac{1}{\epsilon}$)

Example 1. Show that $\lim_{z \to i} \frac{z^2+2}{z-i} = \infty$.

Sol. $f(z) = \frac{z^2+2}{z-i}$

We have to show that $\lim_{z \to i} \frac{1}{f(z)} = 0$

$\lim_{z \to i} \frac{1}{f(z)} = \lim_{z \to i} \frac{z-i}{z^2+2} = \frac{0}{1} = 0$

Example 2. Show that $\lim_{z \to \infty} \frac{z+i}{z+1} = 1$

Sol. $\lim_{z \to \infty} \frac{z+i}{z+1} = \lim_{w \to 0} \frac{\frac{1}{w}+i}{\frac{1}{w}+1}$

$= \lim_{w \to 0} \frac{1+w}{1+w} = 1$

Example 3. Show that $\lim_{z \to \infty} \frac{3iz^3 - 2z}{z^2+3-i} = \infty$

Sol. $\lim_{z \to \infty} \frac{3iz^3 - 2z}{z^2+3-i} = \lim_{z \to \infty} \frac{z^3\left(3i - \frac{2}{z^2}\right)}{z^2\left(1 + \frac{3-i}{z^2}\right)} = \infty$

14. Continuity

Def. (Continuity at a point)
A function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Remark 1. A function f is continuous at a point z_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Example 1. Show that $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is continuous at $z_0 \in \mathbb{C}$.

Sol. $\lim_{z \to z_0} f(z) = f(z_0)$ $\forall z_0 \in \mathbb{C}$.

Def. (Continuity in a region)
A function f is continuous in a region $R \subseteq \mathbb{C}$ if f is continuous at each $z \in R$.

Example 1. A polynomial function

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

is continuous in the region $\mathbb{C}$.

Def. (Continuity in a region)
A function f is continuous in a region $R \subseteq \mathbb{C}$ if f is continuous at each $z \in R$.

Example 1. A polynomial function

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

is continuous in the region $\mathbb{C}$.

Proposition 1. Let f and g be continuous functions at z_0. Then

(a) $f(z) \mp g(z)$ is continuous at z_0.

(b) $f(z)\,g(z)$ is continuous at z_0.

(c) $f(z)/g(z)$ is continuous at z_0, provided that $g(z_0) \neq 0$.

Example 1. The function

$$f(z) = \frac{z^2 - iz + 4}{z}$$

is continuous in the region $\mathbb{C} \setminus \{0\}$.

Proposition 2. Let f be continuous function at z_0 and g be a continuous function at $w_0 = f(z_0)$. Then $g \circ f$ is continuous at z_0.

Proof: $\lim_{z \to z_0} (g \circ f)(z) = (g \circ f)(z_0)$

$\Leftrightarrow \forall \epsilon > 0 \; \exists \delta > 0$ s.t. $|z - z_0| < \delta$

$\Rightarrow |(g \circ f)(z) - (g \circ f)(z_0)| < \epsilon$

Let $\epsilon > 0$ be given and let $\delta =$

g is continuous at $w_0 = f(z_0) \Rightarrow \exists \delta_1 > 0$ such that $|w - w_0| < \delta_1 \Rightarrow |g(w) - g(w_0)| < \epsilon$

f is continuous at $z_0 \Rightarrow \exists \delta_2 > 0$ such that $|z - z_0| < \delta_2 \Rightarrow |f(z) - f(z_0)| < \delta_1$.

Now, Assume $|z - z_0| < \delta = \delta_2$

Then $|w - w_0| = |f(z) - f(z_0)| < \delta_1 \Rightarrow |g(w) - g(w_0)|$

$= |g(f(z)) - g(f(z_0))| < \epsilon$

Proposition 3. If f is continuous at z_0 and $f(z_0) \neq 0$, then $f(z) \neq 0$ for all z in some neighborhood of z_0.

proof:- f is continuous at $z_0 \Rightarrow \forall \epsilon > 0 \ \exists \delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$

Let $N_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta$.

We want to prove that $f(z) \neq 0 \ \forall z \in N_\delta(z_0)$.

Assume $\exists z \in N_\delta(z_0)$ such that $f(z) = 0$

Now

$$z \in N_\delta(z_0) \Rightarrow |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

$$f(z) = 0 \Rightarrow |f(z_0)| < \epsilon = \frac{1}{2}|f(z_0)| \quad C!$$

Remark 2. It follows from Theorem 1 in section 12 that a function $f = u + iv$ is continuous at a point $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

Example 1. The function

$$f(z) = e^x \cos y + i e^x \sin y$$

is continuous everywhere in the complex plane since

$$u(x,y) = e^x \cos y$$

and

$$v(x,y) = e^x \sin y$$

are continuous at each point (x,y) in the plane.

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Proposition 4. If f is continuous in a closed and bounded region R, then there exists a real number $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in R$ and $|f(z)| = M$ for at least one $z \in R$.

15. Derivatives

Def: (The derivative of a function $f(z)$)

(1) Let f be a function and let z_0 be an interior point of the domain of f. The derivative of f at z_0, written $f'(z_0)$ or $\left.\frac{d}{dz} f(z)\right|_{z_0}$, is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided the limit exists.

(2) If $f'(z_0)$ exists, then f is said to be differentiable at z_0.

Example 1. Find $f'(z_0)$ if $f(z) = z^3$.

Sol: $f(z) = z^3$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - z_0^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2 \Delta z + 3z_0 (\Delta z)^2 + (\Delta z)^3 - z_0^3}{\Delta z}$$

$$= 3z_0^2.$$

Example 2. Show that if $f(z) = Re(z)$, then $f'(z)$ does not exist.

Sol:- $f(z) = Re(z)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{Re(z+\Delta z) - Re(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{Re(\Delta z)}{\Delta z} \quad , \Delta z = \Delta x + i\Delta y$$

If $\Delta z = \Delta x$, then $f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$ ①

If $\Delta z = i\Delta y$, then $f'(z) = \lim_{\Delta y \to 0} \frac{0}{i\Delta y} = 0$ ②

① & ② $\Rightarrow f'(z)$ does not exist.

Theorem 1. (Differentiability implies continuity)
If a function f is differentiable at a point z_0, then f is continuous at z_0.

Proof. Assume $f'(z_0)$ exists.
We want to show that $\lim_{z \to z_0} f(z) = f(z_0)$

or $\lim_{z \to z_0} (f(z) - f(z_0)) = 0$

$$\lim_{z \to z_0} (f(z) - f(z_0)) \frac{(z - z_0)}{(z - z_0)} = f'(z_0) \cdot 0 = 0$$

16. Differentiation formulas

Theorem 2. (Differentiation formulas)
Let c be a complex constant, and let f and g be differentiable functions at a point z. Then

(1) $\frac{d}{dz} c = 0$.

(2) $\frac{d}{dz} z^n = n z^{n-1}$, $n \in \mathbb{Z}$

(3) $\frac{d}{dz}[cf(z)] = cf'(z)$.

Ex: $f(z) = |z|^2$ is continuous at each $z \in \mathbb{C}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$\Delta z = \Delta x$ $\quad f'(z) = \lim_{\Delta x \to 0} \frac{|z+\Delta x|^2 - |z|^2}{\Delta x}$

$z = x+iy$ $\quad = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + y^2 - (x^2+y^2)}{\Delta x}$

$z+\Delta z = (x+\Delta x)+iy$ $\quad = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x$

$\Delta z = i\Delta y \therefore f'(z) = \lim_{\Delta z \to 0} \frac{|z+i\Delta y|^2 - |z|^2}{i\Delta y}$

$$= \lim_{\Delta y \to 0} \frac{x^2 + (y+\Delta y)^2 - x^2 - y^2}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\Delta y + (\Delta y)^2}{i\Delta y} = \frac{2y}{1} = -2iy$$

If $z \neq 0$ ($(x,y) \neq (\ ,\ )$) $\Rightarrow f(z)$ does not exist.

(2) بنفس الطريقة

(4) $\frac{d}{dz}[f(z) \mp g(z)] = f'(z) \mp g'(z)$

(5) $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$

(6) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$.

(7) $\frac{d}{dz} g(f(z)) = g'(f(z))f'(z)$

Example 1. Find $f'(z)$ if $f(z) = \frac{(1+3z+2iz)^3}{z-i}$.

Sol. $f'(z) = \frac{3(1+3z+2iz)^2(3+4iz)(z-i) - (1+3z+2iz)^3}{(z-i)^2}$

1.7 Cauchy-Riemann equations

In this section, we study the relation between the existence of f' at a point $z_0 = x_0 + iy_0$ and the existence of u_x, u_y, v_x, v_y at (x_0, y_0)

Theorem 3. (Necessary condition for differentiability)
Suppose that $f(z) = u(x,y) + iv(x,y)$
is differentiable at $z_0 = x_0 + iy_0$.
Then u_x, u_y, v_x, v_y exist at (x_0, y_0)
and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0)

Moreover

$$f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0).$$

Proof. Assume $f'(z_0)$ exists

$$\Rightarrow \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z} = f'(z_0)$$

$$= \lim_{\Delta y \to 0} \frac{f(y_0 + \Delta y_0) - f(y_0)}{i\Delta y}$$

(1) $f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - (u(x_0,y_0) + iv(x_0,y_0))}{\Delta x}$$

$$\left(\begin{matrix} z = z_0 + \Delta x \\ x+iy = x_0+iy_0 \end{matrix}\right) = \lim_{\Delta x \to 0} \left\{ \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \right\}$$

$$= u_x(x_0,y_0) + iv(x_0,y_0)$$

② $$f'(z) = \lim_{\Delta y \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{i\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x_0+iy_0+\Delta y) + iv(x_0+iy_0,\Delta y) - [u(x_0,y_0) + iv(x_0,y_0)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x_0,y_0+\Delta y) - v(x_0,y_0)}{\Delta y} - i\,\frac{u(x_0,y_0+\Delta y) - u(x_0,y_0)}{\Delta y} \right]$$

$$f'(z_0) = v_y(x_0,y_0) - u_y(x_0,y_0)$$

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0).$$

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$v_x(x_0,y_0) = -u_y(x_0,y_0).$$

Example 1. We know that $f(z) = z^3$ is differentiable at every $z \in \mathbb{C}$ and $f'(z) = 3z^2$. Let us show that the theorem is applicable.

$$f(z) = z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$u(x,y)$ $\qquad$ $v(x,y)$

$u_x = 3x^2 - 3y^2$ $\qquad$ $v_x = 6xy$

$u_y = -6xy$ $\qquad$ $v_y = 3x^2 - 3y^2$

$$f'(z) = 3z^2 = u_x + iv_x = 3x^2 + 3y^2 + 6xyi = 3[x^3 + y^3 + 2xyi].$$

Remark 1. The Cauchy-Riemann equations are necessary conditions for the differentiability of f at point z. Thus they can be used to find the points at which f is not differentiable.

Example 1. Show that $f(z) = |z|^2$ is not differentiable at any $z \in \mathbb{C} - \{0\}$.

Sol.: $f(z) = |z|^2 = x^2 + y^2$

$\Rightarrow u(x,y) = x^2 + y^2$, $v(x,y) = 0$

$u_x = 2x \neq v_y = 0$ if $x \neq 0$.

$u_y = 2y \neq -v_x = 0$ if $y \neq 0$.

Thus $u_x = v_y$ and $u_y = -v_x$ only at $(0,0)$

That is the Cauchy-Riemann equations are satisfied at point $(x,y) \neq (0,0)$ and hence $f'(z)$ does not exist for all $z \neq 0$.

Example 2. Show that $f(z) = \bar{z}$ is not differentiable at any $z \in \mathbb{C}$.

Sol.: $f(z) = \bar{z} = x - iy$

and $u(x,y) = x$, $v(x,y) = -y$

$u_x = 1$, $v_y = -1$

$\Rightarrow u_x \neq v_y$

The Cauchy-Reimann equations are not satisfied

18. Sufficient condition for differentiability

Satisfying the Cauchy-Reimann equations at a point is a necessary condition for differentiability but it is not sufficient. We need some extra conditions to make it sufficient.

Example 3. Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $(0,0)$, but $f'(0)$ does not exist.

Sol: $f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{\overline{\Delta z}^2}{\Delta z} - 0}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

If $\Delta z = \Delta x$, then

$$f'(0) = \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{(\Delta x)^2} = 1 \quad \text{---} (1)$$

If $\Delta z = \Delta x + i\Delta x$, then

$$f'(0) = \lim \frac{(\Delta x(1-i))^2}{(\Delta x(1+i))^2} = \frac{(1-i)^2}{(1+i)^2}$$

$$= \frac{-2i}{+2i} = -1 \quad (2)$$

(1) $\neq$ (2) $\Rightarrow f'(0)$ does not exist

$$f(z) = \begin{cases} \dfrac{(x+iy)^2}{x+iy} & , (x,y) \neq (0,0) \\ \dfrac{(x+iy)^2}{x^2+y^2} & \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$u(x,y) = \begin{cases} \dfrac{x^2-3xy^2}{x^2+y^2} \\ 0 \end{cases} \quad , \quad v(x,y) = \begin{cases} \dfrac{y^3-3x^2y}{x^2+y^2} \\ 0 \end{cases}$$

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3}{(\Delta x)^2} - 0}{\Delta x} = 1$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(y+\Delta y) - v(y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{(\Delta y)^3}{(\Delta y)^2}}{\Delta y} = 1$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(x,\Delta y) - u(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(\Delta x) - v(0,0)}{\Delta x} = 0$$

Theorem. (Sufficient condition for differentiability)
Let $f = u + iv$ be defined throughout some ϵ-neighborhood $N_\epsilon(z_0)$ of a point $z_0 = x_0 + iy_0$.
Suppose that u_x, u_y, v_x, v_y exist everywhere in $N_\epsilon(z_0)$ and they are continuous at (x_0, y_0).

Then, if

$u_x = v_y$ and $u_y = -v_x$.

at (x_0, y_0), then $f'(z_0)$ exists.

Example 1. Let $f(z) = \sin x \cosh y + i \cos x \sinh y$. Show that $f'(z)$ exists for all $z \in \mathbb{C}$ and find it.

Sol:- $u(x,y) = \sin x \cosh y$, $v(x,y) = \cos x \sinh y$.

$u_x = \cos x \cosh y$, $v_y = \cos x \cosh y$

$u_y = \sin x \sinh y$, $v_x = -\sin x \sinh y$.

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

u_x, u_y, v_x, v_y are exist and continuous for all $(x,y) \in \mathbb{R}^2$, and $u_x = v_y$, $u_y = -v_x$ $\forall (x,y) \in \mathbb{R}^2$.

Then $f'(z)$ exists $\forall z \in \mathbb{C}$

$\therefore f'(z) = u_x(x,y) + v_x(x,y) = \cos x \cosh y - i \sin x \sinh y$.

Example 2. Find the values of the real numbers a and b that make

$f(z) = x^2 + ay^2 + e^x \cos y + i[bxy + e^x \sin y]$

has derivative at every $z \in \mathbb{C}$.

Sol:- $u(x,y) = x^2 + ay^2 + e^x \cos y$

$v(x,y) = bxy + e^x \sin y$

$u_x = 2x + e^x \cos y$, $u_y = 2ay - e^x \sin y$

$v_x = by + e^x \sin y$, $v_y = bx + e^x \cos y$

exists and continuous everywhere on $\mathbb{R}^2$

$f'(z)$ exists $\forall z \in \mathbb{C} \Rightarrow u_x = v_y$ & $u_y = -v_x$ $\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow \begin{cases} 2x + e^x \cos y = bx + e^x \cos y & \Rightarrow b = 2 \\ 2ay - e^x \sin y = -(by + e^x \sin y) & \Rightarrow 2a = -b \\ & a = -1 \end{cases}$

19. Polar Coordinates

For $z_0 \neq 0$, we have $z_0 = r_0 e^{i\theta_0}$ and hence

$$f(z) = u(r,\theta) + i v(r,\theta).$$

Using the chain rule, we find

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad v_r = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta, \quad v_\theta = -v_x r \sin\theta + u_y r \cos\theta$$

If u and v satisfy the Cauchy-Riemann equations, then

$$u_r = \frac{1}{r} v_\theta$$

$$u_\theta = -r v_r$$

Remark. If $f'(z_0)$ exists, then

$$f'(z_0) = e^{-i\theta}\left[u_r(r_0,\theta_0) + i v_r(r_0,\theta_0)\right].$$

Example 1. Show that $f(z) = 1/z$ is differentiable at each $z \in \mathbb{C} - \{0\}$ and $f'(z) = -1/z^2$.

Sol. $z = re^{i\theta}$

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}\left[\cos\theta - i\sin\theta\right]$$

$$u(r,\theta) = \frac{1}{r}\cos\theta, \quad v(r,\theta) = \frac{1}{r}\sin\theta.$$

$$u_r = -\frac{1}{r^2}\cos\theta, \quad v_\theta = -\frac{1}{r}\cos\theta$$

$$v_r = \frac{1}{r^2}\sin\theta, \quad u_\theta = -\frac{1}{r}\sin\theta.$$

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

$$= \frac{1}{r^2}\cos\theta, \quad = \frac{1}{r^2}\sin\theta$$

$f'(z)$ exists $\forall z \in \mathbb{C} - \{0\}$

$$f'(z) = e^{i\theta}\left[u_r + iv_r\right]$$

$$= e^{-i\theta}\left[-\frac{1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta\right]$$

$$= -\frac{1}{r^2}e^{-i\theta}\left[\cos\theta - i\sin\theta\right]$$

$$= -\frac{1}{r^2}e^{-i\theta}e^{-i\theta} = -\frac{1}{r^2}e^{-2i\theta} = -\frac{1}{z^2}.$$

20. Analytic Functions

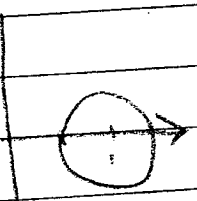
Def. (Analytic Function)

(1) A function f of the complex variable z is analytic at a point z_0 if f is differentiable in some neighborhood of z_0.

(2) A function f of the complex variable z is analytic in an open set if it has a derivative at each point in that set.

Example 1. Show that $f(z) = \frac{1}{z}$ is analytic at $z=1$.

Sol: $f'(z) = -\frac{1}{z^2}$ exists $\forall z \in \mathbb{C}$

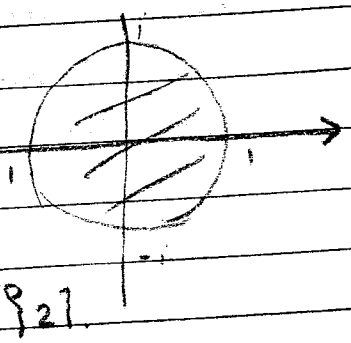


In particular, $f'(z)$ exists $\forall z \in \{z, |z-1| < \frac{1}{2}\} = N_{\frac{1}{2}}(1)$

$\Rightarrow f$ is analytic at $z=1$.

Example 2. Show that $f(z) = \frac{1}{(z-2)^2}$ is analytic in $S = \{z : |z| < 1\}$.

Sol: $f(z) = \frac{1}{(z-2)^2}$, $S = \{z : |z| < 1\}$



S is open

$\Rightarrow f$ is analytic in S if and only if $f'(z)$ exists $\forall z \in S$

$f'(z) = \frac{-2}{(z-2)^3}$ exists $\forall z \in \mathbb{C} - \{2\}$

Then $f'(z)$ exists $\forall z \in S$.
$\therefore$ f is analytic in S.

Example 3. Let $f(z) = |z|^2$

(a) Show that $f'(0) = 0$
(b) Show that $f(z)$ is not analytic at $z = 0$.

Sol:- (a) $f'(0) = 0$

$f(z) = x^2 + y^2$

$\Rightarrow u(x,y) = x^2 + y^2$, $v(x,y) = 0$

$u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$

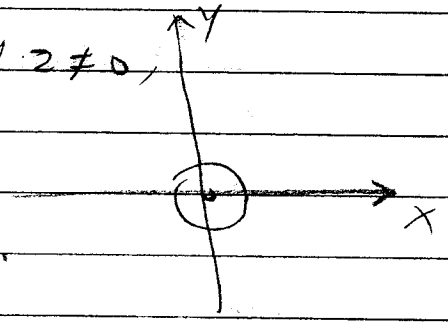
all exists $\forall (x,y) \in \mathbb{R}^2$ and continuous at $(0,0)$.

Moreover $u_x(0,0) = 0 = v_y(0,0)$

$u_y(0,0) = 0 = -v_x(0,0)$

Therefore $f'(0) = u_x(0,0) + i v_x(0,0) = 0$ exists.

(b) Since $f'(z)$ does not exists $\forall z \neq 0$, $f'(z)$ does not exist in any neighborhood of $z = 0$ and hence f it not analytic at $z = 0$.

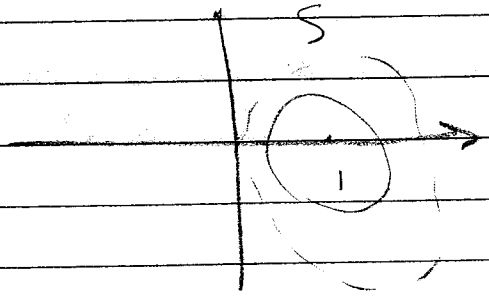


Remark.

(1) The terms regular and holomorphic are also used in the literature to denote analyticity.
(2) A function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S.

Example 1. $f(z) = 1/z$ is analytic in the closed set $S = \{z : |z-1| \leq 1/2\}$ since it is analytic in the open set $\{z : |z-1| < 1\} \supseteq S$.

Sol:-



Def:- (Entire function)

If a function f is analytic at each point in the entire finite complex plane, then f is called an entire function.

Example 1. Every polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is an entire function.

Example 2. Show that $f(z) = \cos x \cosh y - i \sin x \sinh y$ is an entire function

Sol:-

We want to show $f'(z)$ exists $\forall z \in \mathbb{C}$

$u(x,y) = \cos x \cosh y$, $v(x,y) = -\sin x \sinh y$

$u_x = -\sin x \cosh y$, $u_y = \cos x \sinh y$

$v_x = -\cos x \sinh y$, $v_y = -\sin x \cosh y$

all exist and continuous $\forall (x,y) \in \mathbb{R}^2$

Moreover, $u_x = v_y$ and $u_y = -v_x$

Thus $f'(z)$ exist $\forall z \in \mathbb{C}$.

an hence f is entire function.

Def: (Singular point)

If a function f is not analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a singular point, or singularity of f.

Example 1. Show that $z=0$ is a singular point of $f(z)=\frac{1}{z^2}$.

sol: $f(z)=\frac{1}{z^2}$, $f'(z)=\frac{-2}{z^3}$

Example 2. The function $f(z)=|z|^2$ has no singular point since f is nowhere analytic.

Remark. Let D be a domain

(1) If a function f is analytic in D, then f is continuous in D

(2) If $f=u+iv$ is analytic in D, then u and v satisfy the Cauchy-Riemann equations.

(3) If u_x, u_y, v_x, v_y are continuous on D and $u_x=v_y$, $u_y=-v_x$, then f is analytic on D.

(4) If f and g are analytic functions in D, then $f\pm g$, fg are analytic in D and f/g is analytic provided that $g(z)\neq 0$ for all $z\in D$.

(5) A composition of two analytic functions is analytic.

(6) If $f(z) = \frac{P(z)}{Q(z)}$ is a rational function, then the only singular points of f are z such that $Q(z) = 0$.

Example 1. Find the singular points of

$$f(z) = \frac{z}{z^2+1}.$$

Sol:- Singular points are z such that $z^2+1 = 0$

$\Rightarrow z = \pm i$

Example 2. Show that

$$f(z) = z e^x e^{iy}$$

is an entire function.

Sol:- $f(z) = z e^x e^{iy}$

$= e^x (x+iy)(\cos y + i \sin y)$

$= e^x (x\cos y + ix\sin y + iy\cos y - y\sin y)$

$\Rightarrow u(x,y) = e^x (x\cos y - y\sin y)$, $v(x,y) = e^x (x\sin y + y\cos y)$

$u_x = e^x (x\cos y - y\sin y) + e^x \cos y$,

$u_y = e^x (-x\sin y - \sin y - y\cos y)$

$v_x = e^x (x\sin y + y\cos y) + e^x \sin y$

$v_y = e^x (x\cos y + \cos y - y\sin y)$

all exists and continuous for all $(x,y) \in \mathbb{R}^2$

$u_x = e^x [x\cos y - y\sin y + \cos y] = v_y$

$u_y = -e^x (x\sin y + \sin y + y\cos y] = -v_x \quad \forall (x,y) \in \mathbb{R}^2$

Therfore $f(z)$ is an entire function.

Example 3. Show that

$$f(z) = \bar{e}^{y} e^{2xi} \text{ is nowhere analytic.}$$

Sol:- $f(z) = \bar{e}^{y} e^{2xi} = \bar{e}^{y}(\cos(2x) + i\sin(2x))$

$u(x,y) = \bar{e}^{y}\cos 2x$, $v(x,y) = \bar{e}^{y}\sin(2x)$

$u_x = -2\bar{e}^{y}\sin(2x)$, $v_x = 2\bar{e}^{y}\cos(2x)$

$u_y = \bar{e}^{y}\cos(2x)$, $v_y = \bar{e}^{y}\sin(2x)$.

$u_x = v_y \Rightarrow -2\bar{e}^{y}\sin(2x) = \bar{e}^{y}\sin(2x)$

$\Rightarrow 3\sin(2x) = 0 \Rightarrow \sin(2x) = 0$

$\Rightarrow 2x = n\pi$, $n \in \mathbb{Z}$.

$\Rightarrow x = \frac{n\pi}{2}$, $n \in \mathbb{Z}$.

$u_y = -v_x \Rightarrow \bar{e}^{y}\cos(2x) = -2\bar{e}^{y}\cos(2x)$

$\Rightarrow \cos(2x) = 0$

$x = \frac{n\pi}{2} \Rightarrow \cos(n\pi) = 0$

$\Rightarrow (-1)^n = 0$ impossible

Thus the Cauchy-Rieman equations are not satisfied at any $(x,y) \in \mathbb{R}^2$

$\Rightarrow f'(z)$ does not exist for any $z \in \mathbb{C}$

$\Rightarrow f$ is nowhere analytic.

Theorem. If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Proof: Assume $f'(z) = 0 \ \forall z \in D$

$\Rightarrow f'(z) = u_x + iv_x = 0$

$\Rightarrow u_x = 0 = v_x$ —— ① $\forall (x,y) \in D$

Moreover $u_x = v_y \Rightarrow v_y = 0 \ \forall (x,y) \in D$ — (3)

$u_y = -v_x \Rightarrow u_y = 0 \ \forall (x,y) \in D$ — (4)

(1) $u_x = 0 \Rightarrow u(x,y) = g(y)$ for some function g.

(4) $\Rightarrow u_y = g'(y) = 0 \Rightarrow g(y) = c_1 =$ constant.

$\Rightarrow u(x,y) = c_1$

(2) $\Rightarrow v_x = 0 \Rightarrow v(x,y) = h(y)$ for some function h

(3) $\Rightarrow v_y = h'(y) = 0 \Rightarrow h(y) = c_2 =$ constant.

$\Rightarrow v(x,y) = c_2$

Therefore $f(z) = u + iv$

$= c_1 + ic_2$.

Example: 1. Show that if $f(z) = u + iv$ and $\overline{f(z)} = u - iv$ are both analytic in a domain D, then $f(z) =$ constant.

Sol: (1) $f(z) = u + iv$ is analytic.

$\Rightarrow$ $u_x = v_y$, $u_y = -v_x$ on D

(1) (2)

(2) $f(z) = u - iv$ is analytic on D

$\Rightarrow u_x = (-v)_y \Rightarrow u_x = -v_y$ — (3)

$u_y = -(-v)_x \Rightarrow u_y = v_x$ — (4)

(1) & (2) $\Rightarrow v_y = -v_y$

$\Rightarrow 2v_y = 0$

$\Rightarrow v_y = 0$

$\Rightarrow u_x = 0$

(2) & (4) $v_x = -v_x$

$\Rightarrow v_x = 0$

Thus $f'(z) = u_x + iv_x = 0$

By $f(z) =$ constant.

Harmonic Functions

A real-valued function H of two real variables x and y is said to be harmonic in a given domain of the xy-plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x,y) + H_{yy}(x,y) = 0 \qquad (1)$$

known as Laplace's equation

Example (1): It is easy to verify that the function $T(x,y) = e^{y}\sin x$ is harmonic in any domain of the xy-plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig (1)

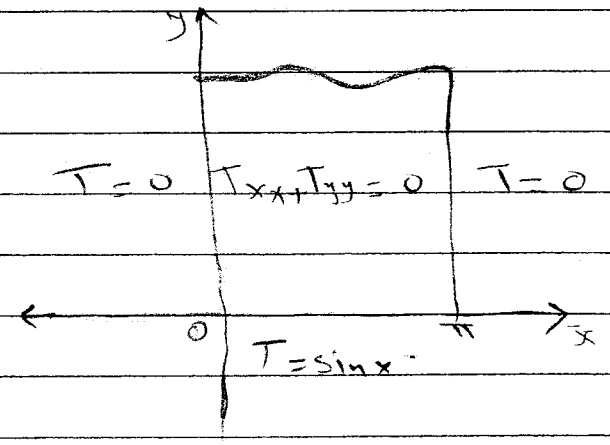


Fig (1)

$$T_{xx}(x,y) + T_{yy}(x,y) = 0$$
$$T(0,y) = 0, \quad T(\pi,y) = 0$$
$$T(x,0) = \sin x, \quad \lim_{y\to\infty} T(x,y) = 0$$

Which describe steady temperatures $T(x,y)$ in a thin

homogenous plate in the xy-plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

Theorem (1): If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component function u and v are harmonic D.

Proof:- Assuming that f is analytic in D, we start with the observation that the first-order partial derivatives of its component functions must satisfy the Cauchy-Rieman equations throughout D:

$$u_x = v_y, \quad u_y = -v_x. \tag{2}$$

Differentiating both sides of these equations with respect to x, we have

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}. \tag{3}$$

Likewise, differentiation with respect to y yields

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}. \tag{4}$$

Now, by a theorem in advanced calculus the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4).

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

Example (2): The function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire, as is shown in exercise (c), Sec. 25. Hence its real component, which is the temperature function $T(x,y) = e^{-y}\sin x$ in Example (1), must be harmonic in every domain of the xy-plane.

Example (3): Since the function $f(z) = i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i}{z^2}\cdot\frac{\bar{z}^2}{\bar{z}^2} = \frac{i\bar{z}^2}{(z\bar{z})^2} = \frac{i\bar{z}^2}{|z|^4} = \frac{2xy + i(x^2-y^2)}{(x^2+y^2)^2}$$

the two functions

$$u(x,y) = \frac{2xy}{(x^2+y^2)^2} \quad \text{and} \quad v(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

are harmonic throughout any domain in the xy-plane that does not contain the origin.

Definition: If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equations (2) throughout D, then v is said to be a harmonic conjugate of u.

Theorem (2): A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

The proof is easy. If v is a harmonic conjugate of u in D, the theorem in Sec. 22 tells us that f is analytic in D. Conversely, if f is analytic in D, we know from theorem (1) that u and v are harmonic in D;

Furthermore, in view of the theorem in Sec 21, the Cauchy-Riemann equations are satisfied.

Example (4): Suppose that

$$u(x,y) = x^2 - y^2 \quad \text{and} \quad v(x,y) = 2xy.$$

Since these are the real and imaginary components, respectively, of the entire function $f(z) = z^2$, we know that v is harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, as verified in Exercise 2(b), Sec 25, the function $2xy + i(x^2 - y^2)$ is not analytic anywhere.

Example (5): We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x,y) = y^3 - 3x^2y \tag{5}$$

is readily seen to be harmonic throughout the entire xy-plane. Since a harmonic conjugate $v(x,y)$ is related to $u(x,y)$ by means of the Cauchy-Reimann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x,y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x,y) = -3xy^2 + \phi(x) \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x,y) = -3xy^2 + x^3 + C \qquad (8)$$

is a harmonic conjugate of $u(x,y)$.

The corresponding conjugate of $u(x,y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C). \qquad (9)$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

The exponential function + Trigonometric function

Chapter 3

Elmentry Functions

In this chapter, we will consider elementry functions of a complex variable. We will introduce complex exponential, trigonometric, hyperbolic, and logarithmic functions.

23. The exponential function:-

We want to define an exponential function of a complex variable $z = x+iy$. We require that

- $f(x) = e^x$ for all $x \in \mathbb{R}$.
- $f'(z) = f(z)$.

Def:- (The exponential function)

We define the exponential function of a complex variable $z = x+iy$ to be

$$e^z = e^x(\cos y + i \sin y)$$

Remark:

1. The domain of the exponential function is the entire complex plane.

2. We sometimes use $\exp(z)$ to denote e^z.

3. When $z = iy$, we have the Euler's formula

$$e^{iy} = \cos y + i \sin y.$$

Example: 1) Express $e^{-1+\frac{\pi}{2}i}$ in the form $a+ib$.

sol: $e^{-1+\frac{\pi}{2}i} = \exp(-1+\frac{\pi}{2}i) = e^{-1}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= \frac{i}{e}$.

Remark: When $z = \frac{1}{n}$, $n = 2, 3, \ldots$, then $e^{\frac{1}{n}} = \sqrt[n]{e}$.

That is $e^{\frac{1}{n}}$ is not the set of all n-th roots of e.

Proposition: (Properties of e^z)

Let $z = x+iy$ Then we have the following properties of z:

1. $|e^z| = e^x$.
2. $\arg(e^z) = y + 2n\pi$, $n \in \mathbb{Z}$.
3. $e^z \neq 0$ for all $z \in \mathbb{C}$.
4. $e^{z_1} e^{z_2} = e^{z_1+z_2}$,
5. $\frac{1}{e^z} = e^{-z}$

6- $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

7- $(e^z)^n = e^{nz}$ for all $n \in \mathbb{Z}$

8- e^z is a periodic function with period of $2\pi i$. That is

$$e^{z+2\pi i} = e^z.$$

Example 1):- Slove the equation $e^z = 1$.

Sol:- 1) $e^z = e^x(\cos y + i \sin y) = 1$

$e^x \cos y = 1$, $e^x \sin y = 0 \Rightarrow \sin y = 0$

$y = n\pi$, $n \in \mathbb{Z}$.

$\Rightarrow e^x \cos n\pi = 1$

$\Rightarrow e^x (-1)^n = 1 \Rightarrow e^x = (-1)^n$, $z = 2m\pi i$, $m \in \mathbb{Z}$.

$e^x > 0 \Rightarrow n = 2m$, $m \in \mathbb{Z}$.

$e^x = 1 \Rightarrow x = 0$, $y = 2m\pi$, $m \in \mathbb{Z}$.

Example:- 2) Find all the value of z such that $e^z = -2$.

Sol:- 2) $-2 = |-2| e^{i\,\mathrm{Arg}(-2)}$

$-2 = 2e^{i\pi}$

$e^{iz} = -2 \Rightarrow e^{xi - y} = 2e^{i\pi}$

$e^{-y} e^{xi} = 2e^{i\pi}$

$\Rightarrow e^{-y} = 2$ and $x = \pi + 2n\pi$, $n \in \mathbb{Z}$.

$\Rightarrow y = -\ln 2$

$z = (2n+1)\pi - i \ln 2$, $n \in \mathbb{Z}$.

24. Trigonometric function

Euler's formula gives

$$e^{iz} = \cos x + i \sin x$$

and

$$e^{-iz} = \cos x - i \sin x$$

Thus we have

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

and

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Def: (The sine and cosine functions)
We define the sine and cosine function of a complex variable z to be

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

and

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Ex: 1) Find $\sin(i\pi) = \frac{1}{2i}(e^{i(i\pi)} - e^{-i(i\pi)})$

$$= \frac{1}{2i}(e^{-\pi} - e^{\pi})$$

$$= i\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = i \sinh \pi.$$

Proposition:-

1. The sine and cosine functions are entire functions

2. $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$

3. $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.

proof:- 1) $\frac{d}{dz}\sin z = \frac{1}{2i}\left[e^{iz} - (-i)e^{-iz}\right]$

$= \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$

2) and 3) H.W

Proposition:- (Trigonometric identities)

Let z_1 and z_2 be complex numbers. Then

(1) $\sin^2 z + \cos^2 z = 1$.

(2) $\sin(z_1 \mp z_2) = \sin z_1 \cos z_2 \mp \sin z_2 \cos z_1$.

(3) $\cos(z_1 \mp z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$.

proof:- 2) $\sin z_1 \cos z_2 = \frac{1}{2i}(e^{iz_1} - e^{-iz_1})\frac{1}{2}(e^{iz_2} + e^{-iz_2})$

$+ \quad = \frac{1}{4i}(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(z_2-z_1)} - e^{-i(z_1+z_2)})$

$\sin z_2 \cos z_1 = \frac{1}{4i}(e^{i(z_2+z_1)} + e^{i(z_2-z_1)} - e^{i(z_1-z_2)} - e^{-i(z_2+z_1)})$.

$= \frac{1}{4i}\left[2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)}\right]$

$=$ □

Proposition:-

(1) If $z = iy$ is pure imaginary, then

$$\sin(iy) = i\sinh y$$

and

$$\cos(iy) = \cosh y$$

(2) If $z = x + iy$, then

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

(3) If $z = x + iy$, then

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:- 1) $\sin(iy) = \frac{1}{2i}\left[e^{i(iy)} - e^{-i(iy)}\right]$

$$= \frac{-i}{2}\left[e^{-y} - e^{y}\right]$$

$$= \frac{i}{2}\left(e^{y} - e^{-y}\right)$$

$$= i \sinh y$$

$$\cosh y = \frac{1}{2}\left(e^{-y} + e^{y}\right)$$

$$\cos(iy) = \cos(x + iy)$$

$$= \cos x \cos(iy) - \sin x \sin(iy)$$

$$= \cos x \cosh y - i \sinh y \sin x$$

Example 1) Find the roots of $\sin z = 1$

Sol:- $\sin z = 1 \Rightarrow z = ??$

$$\sin z = \cos x \cosh y + i \cos x \sinh y = 1$$
$$\sin x \cosh y = 1 \quad \text{and} \quad \cos x \sinh y = 0$$

$x = (2n+1)\frac{\pi}{2} \Rightarrow$ $\Rightarrow \cos x$ or $\sinh y = 0$

$\sin (2n+1)\frac{\pi}{2} \cosh y = 1$ $\Rightarrow x = (2n+1)\frac{\pi}{2}$ or $y = 0, n \in \mathbb{Z}$.

$(-1)^n \cosh y = 1$

and $\cosh y = (-1)^n$

$\cosh y \geq 1 \Rightarrow n = 2m, \ m \in \mathbb{Z}$.

$\cosh y = 1 \Rightarrow y = 0, \ x = (2m+1)\frac{\pi}{2}, \ m \in \mathbb{Z}$

The solutions are

$z = (2m+1)\frac{\pi}{2}, \ m \in \mathbb{Z}$.

$y = 0 \quad \sin x = 1$

$x = (4m+1)\frac{\pi}{2}, \ m \in \mathbb{Z}$.

Example 2):- Show that $\overline{\cos(iz)} = \cos(i\bar{z})$

Sol:- $\cos(iz) = \frac{1}{2}\left[e^{i(iz)} + e^{-i(iz)}\right]$

$$= \frac{1}{2}\left[e^{-z} + e^{z}\right.$$

$$\Rightarrow \overline{\cos(iz)} = \overline{\frac{1}{2}\left[e^{-z} + e^{z}\right]}$$

$$= \frac{1}{2}\left[\overline{e^{-z}} + \overline{e^{z}}\right]$$

$\overline{\cos(iz)} = \frac{1}{2}\left[e^{-\bar{z}} + e^{\bar{z}}\right]$ (نثبتها مباشرة من التعريف) — ①

$\cos(i\bar{z}) = \frac{1}{2}\left[e^{i(i\bar{z})} + e^{-i(i\bar{z})}\right) = \frac{1}{2}\left[e^{-\bar{z}} + e^{\bar{z}}\right]$ — ②

From ② we have

$$\overline{\cos(iz)} = \cos(i\bar{z})$$

Example 31: Find all the roots of $\sin z = \cos z$.

Sol:- $\sin z = \cos z \Rightarrow$

$$\sin x\cosh y + i\cos x\cosh y = \cos x\cosh y - i\sin x\sinh y$$

$$\Rightarrow \sin x\cosh y = \cos x\cosh y \quad \text{---} ①$$

and $\cos x\sinh y = -\sin x\sinh y$ ---②

From ① we have

$$\cosh y(\sin x - \cos x) = 0$$

$\cosh y \geq 1 \Rightarrow \sin x = \cos x$

$\cos x \neq 0$, $\sin x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2} = m\pi$ that is impossible.

$$\Rightarrow \cos x \neq 0, \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4} + n\pi \quad , n \in \mathbb{Z}.$$

From ②

$$\sinh y[\cos x + \sin x] = 0$$

$$\Rightarrow \sinh y\left[\cos\left(\frac{\pi}{4} + n\pi\right) + \sin\left(\frac{\pi}{4} + n\pi\right)\right]$$

$$\Rightarrow \sinh y = \left[\frac{(-1)^n}{\sqrt{2}} + \frac{(-1)^n}{\sqrt{2}}\right] = 0$$

$$\Rightarrow \sinh y = 0 \Rightarrow y = 0$$

Thus the solutn are

$$z = x + iy = \left(\frac{\pi}{4} + n\pi\right) + i0 \quad , n \in \mathbb{Z}.$$

Proposition:- Let $z = x+iy$. Then,

(1) $\sin z^2 = \sin^2 x + \sinh^2 y$
(2) $(\cos z)^2 = \cos^2 x + \sinh^2 y$
(3) $\sin z = 0$ if and only if $z = n\pi$, $n \in \mathbb{Z}$.
(4) $\cos z = 0$ if and only if $z = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

Proof:- H.W

Example 1): Show that $|\cos z| \geq |\cos x|$

Sol:- $(\cos z)^2 = \cos^2 x + \sinh^2 y$

$\geq \cos^2 x$

$\Rightarrow \sqrt{(\cos z)^2} \geq \sqrt{\cos^2 x}$

$\Rightarrow |\cos z| \geq |\cos x|$

The tangent, cantangent, secant and cosecant functions

Def:- We define the tangent, cantangent, secant and cosecant functions of a complex variable z to be

$$\tan z = \frac{\sin z}{\cos z}$$

$$\cot z = \frac{\cos z}{\sin z}.$$

$$\sec z = \frac{1}{\cos z}$$

$$\csc z = \frac{1}{\sin z}.$$

Proposition:-

(1) $\tan z$ and $\sec z$ are analytic everywhere except at the singular points $z = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

(2) $\cot z$ and $\csc z$ are analytic everywhere except at the singular point $z = n\pi$, $n \in \mathbb{Z}$.

(3) $\frac{d}{dz}\tan z = \sec^2 z$, $\frac{d}{dz}\cot z = -\csc^2 z$.

$\frac{d}{dz}\sec z = \sec z \csc z$, $\frac{d}{dz}\csc z = -\csc z \cot z$.

Proof:- H.W

Example 1):- Show that $\tan(z+\pi) = \tan z$.

Sol:- $$\tan(z+\pi) = \frac{\sin(z+\pi)}{\cos(z+\pi)}$$

$$= \frac{\sin z \cos(\pi) - \sin\pi \cos z}{\cos z \cos\pi - \sin z \sin\pi}$$

$$= \frac{-\sin z}{-\cos z} = \tan z$$

25. Hyperbolic functions

Def:- We define the hyperbolic functions of a complex variable z as follows

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

and

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth = \frac{\cosh z}{\sinh z}$$

Proposition:-

1- $\cosh z = \cos(iz)$ and $\sinh z = i\sin(iz)$

2- $\cosh z$ and $\sinh z$ are entire functions

3- $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$

4- $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$

5- $\cosh^2 z - \sinh^2 z = 1$.

Proof. (5) $\cosh^2 x - \sinh^2 x = \cos^2(iz) - (-i(\sin iz))^2$

$= \cos^2 iz + \sin^2 iz = 1.$

6- $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \sinh z_2 \cosh z_1.$

7- $\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$

8- $|\sinh z|^2 = \sinh^2 z + \sin^2 y.$

9- $|\cosh z|^2 = \sinh^2 x + \cos^2 y.$

Proof- (8) $\sinh z = -i\sin(iz)$

$\Rightarrow |\sinh z|^2 = |-i\sin(iz)|^2$

$= |-i|^2 |\sin(iz)|^2$

$= |\sin(-y+ix)|^2$

$$\boxed{\begin{aligned}(\sin z)^2 &= (\sin(x+iy))^2 \\ &= \sin^2 x + \sinh^2 y\end{aligned}} \quad (*)$$

با استخدام (*) ~~سيصبح~~

$= \sin^2(-y) + \sinh^2 x$

$= \sin^2 y + \sinh^2 x.$

10- $\sinh z = 0$ if and only if $z = n\pi i, n \in \mathbb{Z}.$

11- $\cosh z = 0$ if and only if $z = (2n+1)\frac{\pi i}{2}, n \in \mathbb{Z}.$

Proof- (10) $\sinh z = 0 \iff -i\sin(iz) = 0$

$\iff \sin(iz) = 0 \iff iz = n\pi$

$\iff z = -n\pi i$

Example 1) Show that $\sinh(z+i\pi) = -\sinh z$.

Sol:- $\sinh(z+i\pi) = -i\sin(iz-\pi)$

$= -i(\sin(iz)\cos\pi - \cancel{\cos(iz)\sin\pi})$

$= i\sin(iz)$

$= -\sinh z$.

Example 2) Find all roots of the equation $\cosh z = i$

$$z = \ln|1 \pm \sqrt{2}| + (\pm\frac{\pi}{2} + 2n\pi)i, \quad n \in \mathbb{Z}.$$

Sol:- $\cosh z = \cos(iz) \overset{?}{=} i$

$\Rightarrow \cosh(-y+ix) = 0$

$\Rightarrow \cos y \cosh x - i\sin(-y)\sinh x = i$

$\cos y \cosh x + i \sin y \sinh x = i$

$\Rightarrow \boxed{\cos y \cosh x = 0}$ —①, $\boxed{\sin y \sinh x = 1}$ ②

from ① $\cos y = 0$, $\cosh x \neq 0$ (ابدا) $\Rightarrow y = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

substite in ②

$\sin(2n+1)\frac{\pi}{2} \sinh x = 1$

$\Rightarrow (-1)^n \sinh x = 1$

$\Rightarrow \sinh x = (-1)^n$

$\Rightarrow \frac{1}{2}[e^x - e^{-x}] = (-1)^n$

$\cosh x \neq 0$

$\cosh x \neq$ سالب

لذلك العدد يجب ان زوجي

وليس فردي

$\sinh x =$ ممكن سالب او موجب

ضربنا بـ e^{2x}

$\Rightarrow e^{2x} - 1 = 2(-1)^n e^x$

$\Rightarrow (e^x)^2 - 2(-1)^n e^x - 1 = 0$

$\Rightarrow e^x = \frac{2(-1)^n \mp \sqrt{4+4}}{2}$

$= (-1)^n \mp \sqrt{2}$

$e^x > 0 \Rightarrow$

$e^x = (-1)^n + \sqrt{2}$

$x = \ln\left[(-1)^n + \sqrt{2}\right]$

The solution are

$z = \ln\left[(-1)^n + \sqrt{2}\right] + (2n+1)\frac{\pi i}{2}$, $n \in \mathbb{Z}$

Log

The logarithmic and its branches

$\omega = \log z = ??$, $e^{\omega} = z \quad (z \neq 0)$

let $\omega = u + iv$, $z = |z| e^{i \operatorname{Arg} z}$

$\Rightarrow e^{\omega} = e^{u} e^{iv} = |z| e^{i \operatorname{Arg} z}$

$\Rightarrow |z|$ and $v = \operatorname{Arg} z + 2n\pi$, $n \in \mathbb{Z}$

$\Downarrow$

$u = \ln |z| = 1$

$\therefore \omega = \log z = \ln|z| + i(\operatorname{Arg} z + 2n\pi)$, $n \in \mathbb{Z}$

The logarithmic function and its branches.

In this section, we will define the logarithmic function of a complex variable. As a first step we solve the equation $e^{\omega} = z$, where z is a nonzero complex number. Writing $\omega = u + iv$ and $z = |z| e^{i \operatorname{Arg} z}$, we have the solutions

$\omega = \ln|z| + i(\operatorname{Arg} z + 2n\pi)$, $n \in \mathbb{Z}$.

1) Def:- (The logarithm of a complex number)
we define the logarithm of a non zero complex number z to be

$\log z = \ln|z| + i(\operatorname{Arg} z + 2n\pi)$, $n \in \mathbb{Z}$.

or

$\log z = \ln|z| + i \arg z$.

Note:- $\ln 1 = 0$, but $\log 1 = \ln|1| + i \arg 1$
$= i 2n\pi$, $n \in \mathbb{Z}$

Log

Example 1): Find $\log(-1)$ and $\log(1+i)$

Sol: $\log(-1) = \ln|-1| + i\arg(-1)$

$= i(\pi + 2n\pi) = (2n+1)\pi i$, $n \in \mathbb{Z}$

$\log(1+i) = \ln|1+i| + i\arg(1+i)$

$|1+i| = \sqrt{2}$, $\arg(1+i) = \text{Arg}(1+i) + 2n\pi$, $n \in \mathbb{Z}$

$= \frac{\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$.

$\therefore \log(1+i) = \ln\sqrt{2} + (2n + \frac{1}{4})\pi i$, $n \in \mathbb{Z}$

Example 2): Find $\log(e^2)$

Sol: $\log e^2 = \ln|e^2| + i\arg e^2$

$= \ln(e^x) + i(y + 2n\pi)$

$= x + iy + 2n\pi i$

$= 2 + 2n\pi i$, $n \in \mathbb{Z}$.

$e^z = e^x(\cos y + i\sin y)$

$|e^z| = \sqrt{e^{x^2}(\cos^2 + \sin^2 y}$

$|e^z| = e^x$.

Remark:-

1- $\log z$ is not a function. We will call it a multiple-valued function

2- $e^{\log z} = z$ for all $z \neq 0$.

3- $\log e^z = z + 2n\pi i$, $n \in \mathbb{Z}$, for all $z \in \mathbb{C}$

Def. The principal value of the logarithm of a nonzero complex number z, $\log z$, is defined to be

$$\text{Log } z = \ln|z| + i \text{Arg } z.$$

Example 1) Find $\text{Log}(1+\sqrt{3}i)$.

Sol.: $\text{Log}(1+\sqrt{3}i) = \ln|1+\sqrt{3}i| + i \text{Arg}(1+\sqrt{3}i)$

$$= \ln 2 + i \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\therefore \text{Log}(1+\sqrt{3}i) = \ln 2 + i\frac{\pi}{3}$$

Remark :-

1- $\text{Log } z$ is a function.

2- $\log z = \text{Log } z + 2n\pi i$, $n \in \mathbb{Z}$.

Branches of $\log z$

As we mentioned above, $\log z$ is a multiple-valued funct... Thus to define a single-valued function we restrict the domain and define functions as

$$\log z = \ln|z| + i \arg z, \quad \alpha < \arg z \leq \alpha + 2\pi$$

where α is any real number. However these functions are not continuous in their domains. To see this let us consider $\text{Log } z$. If we approach -1 along the curve

$$z = e^{(\pi - t)i}, \quad 0 \leq t \leq \pi$$

then $\lim_{z \to -1} \text{Log } z = \lim_{t \to 0} (\pi - t)i = \pi i$.

But if we approach -1 along the curve

Log

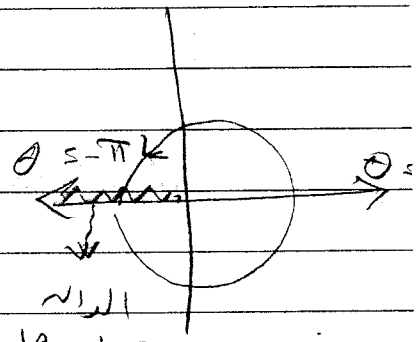
$$z = e^{-(\pi - t)i}, \quad 0 \le t \le \pi;$$

then

$$\lim_{z \to -1} \text{Log } z = \lim_{t \to 0} -(\pi - t)i = -\pi i$$

But if we approach -1 along the curve

$$z = e^{-(\pi - t)i}, \quad 0 \le t \le \pi.$$



then

$$\lim_{z \to -1} \text{Log } z = \lim_{t \to 0} -(\pi - t)i = -\pi i$$

Thus $\lim_{z \to -1} \text{Log } z$ does not exist and hence Log z is not continuous at -1.

The same arguments can be used to show that Log z is not continuous at any negative real number. Deleting these points from the domain of Log z, we obtain an analytic function

$$\text{Log } z = \ln|z| + i \text{Arg } z, \quad |z| > 0, \quad -\pi < \text{Arg } z < \pi.$$

Def. Let f be a multiple-valued function. A single-valued function F is called a branch of f if there is a domain D such that

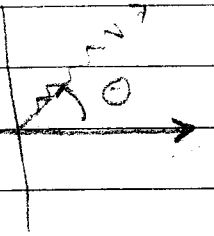
1. $F(z) \in f(z)$ for all $z \in D$.
2. F is analytic in D.

5/ Log

Example 1) $\text{Log } z = \ln|z| + i \operatorname{Arg} z$, $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$.
is a branch of $\log z$. It is called the principle branch.

Prop:- For any real number α.
$\log z = \ln|z| + i \arg z$, $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$
is a branch of $\log z$ and

$$\frac{d}{dz} \log z = \frac{1}{z}, \quad |z| > 0, \quad \alpha < \arg z < \alpha + 2\pi$$



proof:- $\log z = \ln|z| + i \arg z$
let $z = re^{i\theta}$ $\quad \alpha < \theta < \alpha + 2\pi$
$\log z = \ln r + i\theta = u + iv$
$\Rightarrow u = \ln r, \quad v = \theta$
$u_r = \frac{1}{r} v_\theta \Rightarrow \frac{1}{r} = \frac{1}{r}$

$v_r = \frac{1}{r} v_\theta \Rightarrow 0 = 0$

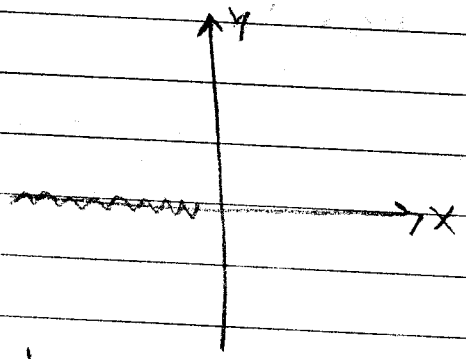
$$\frac{d}{dz} \log z = e^{-i\theta}\left[u_r + iv_r\right] = e^{-i\theta}\left[\frac{1}{r} + i0\right] = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Def (Branch cut and branch point)

1- A branch cut is a portion of a line or curve that is introduced to define a branch of a multiple-valued function.
2- A point that is common to all branch cuts of a function f is called a branch point of f.

Log

Example 1) The branch cuts for the principle branch $\operatorname{Log} z$ is the origin and the ray $\theta = \pi$ and the branch point is the origin.



Example 2) Find the roots of the equation $\log z = 1$.

Sol:- $\log z = 1$

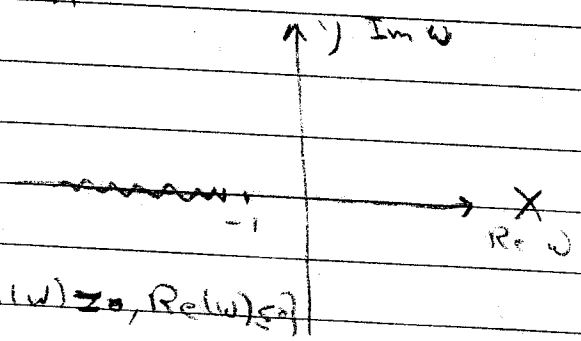
$\Leftrightarrow e^{\log z} = e^{1}$ $\qquad$ $(w = \log z \Leftrightarrow e^{w = \log z} = z$

$\Rightarrow z = e$

Note:- ① $e^{\log z} = z$

② $\log e^{z} = z + 2n\pi i$

Example 3):- Show that $\operatorname{Log}(z+1)$ is analytic everywhere except on the half line $y = 0$, $x \leq -1$.



Sol:- $\operatorname{Log} w = f(w)$

$w = g(z) = z + 1$

$\operatorname{Log}(z+1) = (f \circ g)(z) = f(g(w))$

$\operatorname{Log} w$ is analytic in $\mathbb{C} - \{w : \operatorname{Im}(w) = 0, \operatorname{Re}(w) \leq 0\}$

$g(z) = z + 1$ is entire

$\Rightarrow \operatorname{Log}(z+1) = f(g(z))$ is analytic

if $w = g(z) = z + 1 \in D$

$w = z + 1 = x + iy + 1$

$\operatorname{Im}(w) = y = 0$, $\operatorname{Re}(w) = x + 1 \leq 0 \Rightarrow x \leq -1$.

7.11

Log

Some identities involving logarithms

Many identities involving logarithms of positive real number remain valid, with certain modification, for logarithms of nonzero complex numbers.

Proposition: Let z_1 and z_2 be two nonzero complex numbers. Then

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

Proof. $\log(z_1 z_2) = \ln|z_1 z_2| + i \arg(z_1 z_2)$

$= \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg z_2)$

$= \log z_1 + \log z_2.$

Remark. The statements in the above proposition have the same meaning as the statements for $\arg(z_1 z_2)$ and $\arg(z_1/z_2)$. That is it means that if two of the three logarithms are specified, then there is a third logarithm such that they are true.

Example 1) Let $z_1 = -1$ and $z_2 = i$, Verify that $\log(z_1 z_2) = \log z_1 + \log z_2$.

Sol. $\log(z_1 z_2) = \log(-i)$

$= \ln|-i| + i(\arg(-i) + 2n\pi)$, $n \in \mathbb{Z}$

$= \ln(1) + i\left(2n - \frac{1}{2}\right)\pi$, $n \in \mathbb{Z}$

$= i\left(2n - \frac{1}{2}\right)\pi$, $n \in \mathbb{Z}$

Log

$\log z_1 = \log(-1) = \ln|-1| + i(\operatorname{Arg}(-1) + 2n_1\pi)$, $n_1 \in \mathbb{Z}$

$\log z_1 = i(2n_1+1)\pi$, $n \in \mathbb{Z}$

$\log z_2 = \log(i) = \ln|i| + i(\operatorname{Arg} i + 2n_2\pi)$, $n_2 \in \mathbb{Z}$.

$\log(z_1 z_2) = \log z_1 + \log z_2$

$\Leftrightarrow i\pi(2n - \frac{1}{2}) = \pi i(2n_1+1) + \pi i(2n_2 + \frac{1}{2})\pi$

$\Leftrightarrow 2n = 2n_1 + 2n_2 + 2$

$\Leftrightarrow \boxed{n = n_1 + n_2 + 1}$

Example 2) let $z_1 = -1$ and $z_2 = i$ show that $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$.

Sol: $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(-i) = -\frac{\pi}{2} i$

$\operatorname{Log} z_1 = \operatorname{Log}(-1) = \pi i$

$\operatorname{Log} z_2 = \operatorname{Log}(i) = \frac{\pi}{2} i$

$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \frac{3\pi}{2} i \neq -\frac{\pi}{2} i = \operatorname{Log}(z_1 z_2)$.

Proposition:- Let z be a nonzero complex number Then

$z^n = e^{n \log z}$, $n \in \mathbb{Z}$.

and

$z^{1/n} = e^{\frac{1}{n}\log z}$, $n \in \mathbb{N}$

9//

Log

Proof: ① $z^n = e^{n \log z}$

$e^{n \log z} = e^{n[\ln|z| + i \arg z]} = e^{n[\ln|z| + i(\operatorname{Arg} z + 2k\pi)]}$, $k \in \mathbb{Z}$

$= e^{n \ln|z|} \, e^{in[\operatorname{Arg} z + 2k\pi]}$, $k \in \mathbb{Z}$

$= |z|^n e^{in \operatorname{Arg} z} \cdot e^{i 2kn\pi}$, $e^{i 2kn\pi} = 1$

$= (|z| e^{i \operatorname{Arg} z})^n = z^n$

proof: ② $z^{y_n} = e^{y_n \log z}$, $n \in \mathbb{Z}$

$e^{y_n \log z} = e^{y_n [\ln|z| + i(\operatorname{Arg} z + 2k\pi)]}$, $k \in \mathbb{Z}$

$= e^{y_n \ln|z|} \, e^{i y_n (\operatorname{Arg} z + 2k\pi)}$

$= \sqrt[n]{|z|} \, e^{i(\operatorname{Arg} z + 2k\pi)} = z^{y_n}$

Example: 1) Show that $\operatorname{Log}(i^3) \neq 3 \operatorname{Log} i$.

Sol: $\log z^3 = 3 \log z$

$\operatorname{Log}(i^3) = \operatorname{Log}(-i) = \ln|-i| + i \operatorname{Arg}(-i)$

$= -\frac{\pi}{2} i$

$3 \operatorname{Log} i = 3[\ln|i| + i \operatorname{Arg} i]$

$= 3 \frac{\pi}{2} i$

$\Rightarrow \operatorname{Log}(i) \neq 3 \operatorname{Log} i$.

Complex Exponents

In this section, we will use the logarithms to define complex exponents z^c and e^z

Def. Let z be a nonzero complex number and let c be any complex number. We define z^c as

$$z^c = e^{c \log z}$$

where

$$\log z = \ln|z| + i \arg z$$

is the multiple-valued function.

Example 1) Find the values of i^i.

Sol: $i^i = e^{i \log i}$

$= e^{i[\ln|i| + i \arg(i)]}$

$i^i = e^{-2n\pi}$, $n \in \mathbb{Z}$.

Example 2): Find the values of i^{2+i}

Sol: $i^{2+i} = e^{(2+i)\log i} = e^{(2+i)[\ln|i| + i \arg i]}$

$= e^{(2+i)\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]}$, $n \in \mathbb{Z}$.

$= e^{(4n+1)\pi i - \left(2n + \frac{1}{2}\right)\pi}$

$i^{2+i} = e^{-\left(2n+\frac{1}{2}\right)\pi} e^{(4n+1)\pi i}$

exponents

$$= e^{-(2n+\frac{1}{2})\pi} \; e^{4n\pi i} \; e^{\pi i}$$

$$= e^{-(2n+\frac{1}{2})\pi} \cdot 1 \cdot (-1), \quad n \in Z.$$

$$i^{2+i} = -e^{-(2n+\frac{1}{2})\pi}, \quad n \in Z.$$

Ex:- 3) Find the values of

$$(1+\sqrt{3}i)^{3/2}$$

Sol:- $(1+\sqrt{3}i)^{3/2} = \left[(1+\sqrt{3}i)^3\right]^{1/2}$

$$= e^{3/2 \log(1+\sqrt{3}i)}$$

$$= e^{3/2[\ln|1+\sqrt{3}i| + i \arg(1+\sqrt{3}i)]}$$

$$= e^{3/2[\ln 2 + i(\frac{\pi}{3} + 2n\pi)]}, \quad n \in Z.$$

$$= e^{3/2 \ln 2} \cdot e^{i(\pi/2 + 3n\pi)}$$

$$= \sqrt{2^3} \cdot e^{i\frac{\pi}{2}} \; e^{3n\pi i}$$

$$= \sqrt{8} \cdot i \cdot (-1)^n$$

$$= \mp i\sqrt{8}.$$

Remark:- In general, z^c is a multiple-valued function.

Proposition:- If $z \neq 0$ and c are complex numbers, then

$$\frac{1}{z^c} = z^{-c}.$$

Proof:- H.w

Proposition:- Let α be a real number and let

$$\log z = \ln|z| + i \arg z, \quad |z| > 0, \quad \alpha < \arg z < \alpha + 2\pi$$

be a branch of $\log z$. Then $z^c = e^{c \log z}$ is analytic in

$$D = \{(r,\theta) : r > 0, \ \alpha < \theta < \alpha + 2\pi\}.$$

Moreover

$$\frac{d}{dz} z^c = c z^{c-1}$$

proof:- $\log z = \ln|z| + i \arg z$, is analytic

$D = \{z : |z| > 0, \ \alpha < \arg z < \alpha + 2\pi\}$

e^z is an entire function

$\Rightarrow z^c = e^{c \log z}$ is analytic in D

$$\frac{d}{dz} z^c = \frac{d}{dt} e^{c \log z}$$

$$= e^{c \log z} \cdot \frac{d}{dz}(c \log z)$$

$$= z^c \frac{c}{z} = c z^{c-1}$$

Def. The principal value of z^c is

$$z^c = e^{c \operatorname{Log} z}$$

Example 1) Find the principal value of $(1+i)^{2+3i}$.

Sol:- The principal value of

$$(1+i)^{2+3i} = e^{(2+3i)\operatorname{Log}(1+i)}$$

$$= e^{(2+3i)[\ln|1+i| + i \operatorname{Arg}(1+i)]}$$

$$= e^{(2+3i)(\ln\sqrt{2} + i\frac{\pi}{4})}$$

$$= e^{2\ln\sqrt{2} + i\frac{\pi}{2} + 3i\ln\sqrt{2} - \frac{3\pi}{4}}$$

$$= e^{2\ln\sqrt{2}} \cdot e^{i\frac{\pi}{2}} \cdot e^{3i\ln\sqrt{2}} \cdot e^{-\frac{3\pi}{4}}$$

$$= 2 i e^{-3\pi/4} [\cos(3\ln\sqrt{2}) + i \sin(3\ln\sqrt{2})]$$

The square root function

As a special case of the function z^c we will study $z^{1/2}$.
By definition

$$z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z} = \pm\sqrt{|z|}\, e^{\frac{i}{2}\operatorname{Arg} z}$$

This is a two-valued function.
The principal value of $z^{1/2}$ is

$$z^{1/2} = \sqrt{|z|}\, e^{\frac{i}{2}\operatorname{Arg} z} = \sqrt{|z|}\, e^{\frac{i}{2}\operatorname{Arg} z}$$

$$z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z} = e^{\frac{1}{2}[\ln|z| + i(\operatorname{Arg} z + 2n\pi)]}$$

exponents

$$= \sqrt{|z|}\, e^{\frac{1}{2} i \operatorname{Arg} z}\, e^{in\pi}$$
$$= \pm \sqrt{|z|}\, e^{\frac{1}{2} i \operatorname{Arg} z}$$

$$z^{1/2} = \sqrt{|z|}\, e^{\frac{1}{2} i \arg z} \qquad \alpha < \arg z < \alpha + 2\pi.$$

$$\frac{d}{dz} z^{1/2} = \frac{1}{2} z^{-1/2} = \frac{1}{2 z^{1/2}}.$$

Example:- Find values of $(1+i)^{1/2}$.

Sol:- $(1+i)^{1/2} = \pm \sqrt{|1+i|}\, e^{\frac{1}{2} i \operatorname{Arg}(1+i)}$

$$= \pm \sqrt{\sqrt{2}}\, e^{i \frac{\pi}{8}}$$

$$= \pm \sqrt[4]{2}\, e^{i\pi/8}$$

The function c^z

Def:- Let c be a nonzero complex number and let z be any complex number. We define

$$c^z = e^{z \log c}. \qquad c \neq 0$$

where

$$\log c = \ln |c| + i \arg c$$

is the multiple-valued function.

Example:- Consider 2^z and find 2^i and $2^{(i+1)}$.

Sol:- $2^z \Rightarrow z = i \Rightarrow 2^i = e^{i \log 2}$

$$= e^{i[\ln|2| + i(0 + 2n\pi)]}$$
$$= e^{i \ln 2 - 2n\pi} \qquad n \in \mathbb{Z}$$
$$= e^{i \ln 2}\, e^{-2n\pi}$$

15 exponents

$z = 1+i$

$2^{1+i} = e^{(1+i)\log 2}$

$= e^{(1+i)[\ln 2 + 2n\pi i]}$

$= e^{\ln 2 + 2n\pi i + i\ln 2 - 2n\pi}$

$= 2\, e^{-2n\pi}\, e^{(2n\pi + \ln 2)i}$

Proposition: If a value of $\log c$ is specified the c^z is an entire function and

$$\frac{d}{dz} c^z = c^z \log c$$

proof - H.W

Example: If $\log i = \operatorname{Log} i = \frac{\pi}{2} i$, then

$i^{2z} = e^{(\pi/2) i 2}$ is an entire function.

sol:- $i^{2z} = e^{2z \log i}$

$\log i = \operatorname{Log} i = \frac{\pi}{2} i$

$i^{2z} = e^{(\pi/2) i 2}$

Chapter 4 (Integrals)

Complex-valued function

$w(t)$

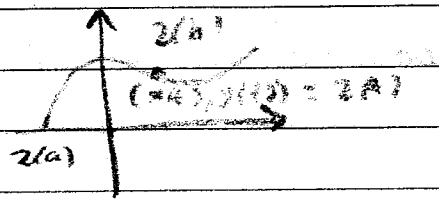
In this chapter, we consider integrations of functions of a complex variable. We will consider line integrals; that is integrations of functions along curves in the complex plane.

In this chapter we find $\int f(z)\,dz$

$$\therefore \int f(z)\,dz = \int [u(x,y) + iv(x,y)](dx + i dy)$$

سنحسب تكامل الدالة المركبة على منحنى

$$\therefore \int f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$



Complex-valued functions $w(t)$:-

In order to study integrals of functions of complex variable, we first consider complex valued functions **of a real variable.**

$$w: D \subset \mathbb{R} \longrightarrow R \subset \mathbb{C}$$

$$w(t) = u(t) + iv(t)$$

Def. 1- Let t be a real variable and let $u(t)$ and $v(t)$ be real-valued functions. Then $w(t) = u(t) + iv(t)$ is complex-valued function of real variable.

2- If u and v are differentiable at a point t, then $w = u + iv$ is differentiable at t and

$$\frac{d}{dt} w(t) = u'(t) + i v'(t).$$

نتعامل مع الاتصال (الاستمرارية) والاشتقاق للدالة $w(t)$ عن طريق
اذا كانت u و v متصلة فان w متصلة واذا كانت u و v قابلة
للاشتقاق فأن $w(t)$ قابلة للاشتقاق.

Ex:- $w = 2t + i\cos t$

$$\Rightarrow w' = 2 - i\sin t$$

Proposition:- Let z_0 be a complex number then

1- $\frac{d}{dt}\left[z_0 w(t)\right] = z_0 w'(t)$

2- $\frac{d}{dt}\left[e^{z_0 t}\right] = z_0 e^{z_0 t}$.

Proof:- ① H.w

Proof: ② Let $z_0 = x_0 + iy_0$

$$e^{tz_0} = e^{tx_0 + iy_0 t}$$

$$= e^{tx_0} e^{iy_0 t}$$

$$= e^{x_0 t}\left[\cos(y_0 t) + i\sin(y_0 t)\right.$$

$$\Rightarrow \frac{d}{dt} e^{tz_0} = \frac{d}{dt}\left[e^{x_0 t}(\cos(y_0 t)\right] + i\frac{d}{dt}\left[e^{x_0 t}\sin(y_0 t)\right]$$

$$= x_0 e^{x_0 t}\cos(y_0 t) - y_0 e^{x_0 t}\sin(y_0 t)$$

$$+ i\left[x_0 e^{x_0 t} + y_0 e^{x_0 t}\cos(y_0 t)\right] \quad \text{---} \quad (1)$$

$$2_0 e^{2_0 t} = e^{x_0 t}(x_0 + iy_0)\left[\cos(y_0 t) + i\sin(y_0 t)\right]$$

$$= \text{---}$$

Def. If $u(t)$ and $v(t)$ are integrable on the interval $[a,b]$, then we say that

$$w(t) = u(t) + iv(t)$$

is integrable on $[a,b]$ and we define

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Example: 1) Find $\int_0^{\pi/6} e^{2ti}\,dt =$

$$\int_0^{\pi/6} e^{2ti}\,dt = \int_0^{\pi/6}\left[\cos(2t) + i\sin(2t)\right]dt$$

$$= \frac{1}{2}\sin(2t)\Big|_0^{\pi/6} + i\left[-\frac{1}{2}\cos(2t)\right]\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{3} - \sin 0\right] - \frac{i}{2}\left[\cos\frac{\pi}{3} - \cos 0\right]$$

$$= \frac{1}{2}\,\frac{\sqrt{3}}{2} - \frac{i}{2}\left[\frac{1}{2} - 1\right]$$

$$= \frac{\sqrt{3}+i}{4}.$$

Remark. The usual rules of integration for real functions remain valid for the integration of complex-valued functions $w(t)$.

Proposition:-

1- Let z_0 be a complex number. Then

$$\int_a^b [z_0 w(t)]\, dt = z_0 \int_a^b w(t)\, dt$$

2- If $W'(t) = w(t)$, then

$$\int_a^b w(t)\, dt = W(b) - W(a)$$

Another solution of Ex. ①

$$\int_0^{\pi/6} e^{2ti}\, dt = \frac{e^{2ti}}{2i}\Big|_0^{\pi/6}$$

Examplex: 1) Find $\int_0^{\infty} e^{-t(1+i)}\, dt$. (improper integral)
Converage or diverage

ملاحظة: تكون $w(t)$ في الكمال Converage اذا كانت $u(t)$
و Converage $v(t)$.

$$\int_0^{\infty} e^{-t(1+i)}\, dt = \lim_{c\to\infty} \int_0^{c} e^{-t(1+i)}\, dt$$

$$= \lim_{c\to\infty} \frac{e^{-t(1+i)}}{-(1+i)}\Big|_0^{c}$$

$$= \frac{-1}{1+i} \lim_{c\to\infty} \left[e^{-c(1+i)} - e^{0}\right]$$

ضربنا بالمرافق

$$= -\frac{(1-i)}{2} \lim_{c\to\infty} \left[e^{-c}(\cos c - i \sin c) - 1\right]$$

دالة زوجية دالة فردية

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* $e^{-\infty} \to 0, \quad -1 < \cos\alpha < 1$

$= \frac{-(1-i)}{2}(0-1)$

$= \frac{1-i}{2}$

Proposition:- If $a \le b$, then $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$

Proof:-

① If $\int_a^b w(t)\,dt = 0$, then

$$\left|\int_a^b w(t)\,dt\right| = 0 \quad \le \quad \int_a^b |w(t)|\,dt \ge 0$$

② If $\int_a^b w(t)\,dt \neq 0$, then

it can be written as

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \quad \text{where}$$

$$r_0 = \left|\int_a^b w(t)\,dt\right|, \quad \theta_0 = \arg\left(\int_a^b w(t)\,dt\right)$$

Now $\left|\int_a^b w(t)\,dt\right| = r_0 = e^{-i\theta_0}\int_a^b w(t)\,dt$.

$$\Rightarrow r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$\Rightarrow Re(r_0) = Re\int_a^b e^{-i\theta_0} w(t)\,dt$$

$$r_0 = \int_a^b Re\left(e^{-i\theta_0} w(t)\right)dt$$

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But $\mathrm{Re}(e^{i\theta_0} w(t)) \le |e^{i\theta_0} w(t)|$

$\le |e^{i\theta_0}||w(t)|$

$\le |w(t)|$

thus

$$v_0 = \left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt$$

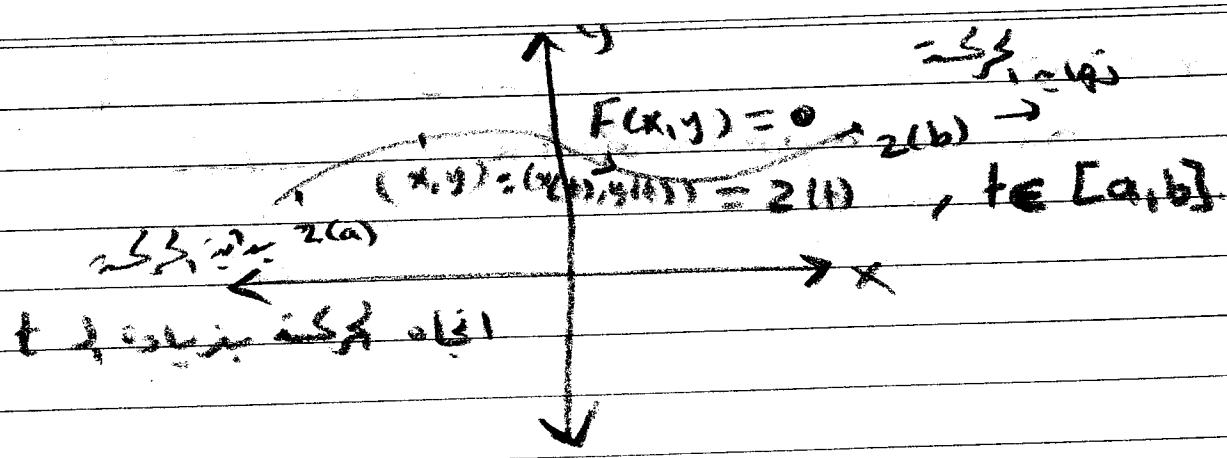
Corollary: If both $\int_a^\infty w(t)\,dt$ and $\int_a^\infty |w(t)|\,dt$ exists, then $\left| \int_a^\infty w(t)\,dt \right| \le \int_a^\infty |w(t)|\,dt$.

Contours

As we mentioned before, we will study integrals of complex-valued functions of complex variable along curves in the complex plane.
In this section, we will study classes of curves that are suitable for such integrals

A curve in the complex plane can be described via the parameterization

$z(t) = x(t) + iy(t) \qquad a \le t \le b.$



الخواص الأساسية لهذه المنحنيات الحسنى (Countor) للتعبير عن هذه المنحنيات بعدة طرق بـ $F(x,y)=0$ مثل دائرة او قطع زائد او باستخدام Poloar form او parametzation وهي سوف نعتمدها حيث المنحنى يتكون من نقاط كل نقطة هي (x,y) سأضع علاقة تعبر عن كل النقاط عن طريق وضع وسيط حيث $(x,y)=(x(t),y(t))$ فلو مثلنا هناك جسم يتحرك على المنحنى فعند كل زمن يكون الجسم بموقع معين، اتجاه الحركة سيكون باتجاه زيادة t (الزمن)

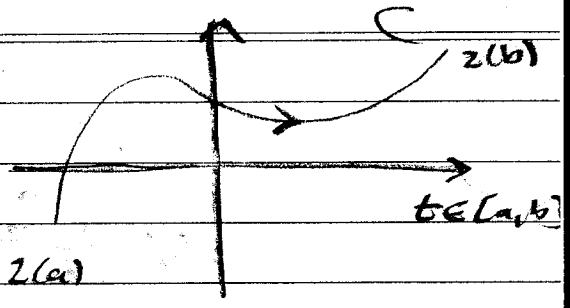
Def. Let C be a curve in the complex plane represented by

$$z(t) = x(t) + i\,y(t) \quad , t \in [a,b]$$

(1) If $x(t)$ and $y(t)$ are cont. functions of t, then C is called an arc or cont. curve

(2) If $x'(t)$ and $y'(t)$ exist and cont. for all $t \in [a,b]$ then C is called a differentiable arc.

Example:- 1) $z(t) = t + 2it^2$, $t \in [-1, 2]$ is an arc, a portion of the parabola $y = x^2$, it is a differentiable arc

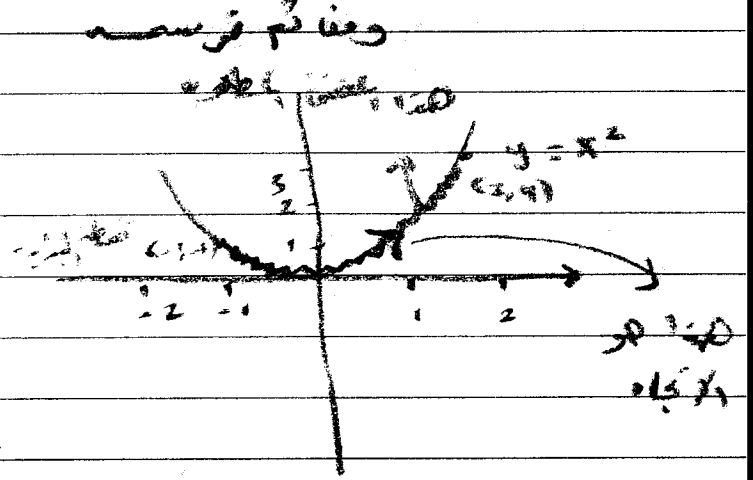


$x(t) = t$, $y(t) = t^2$

↓ Cont. ↓ Cont.

∴ $z(t)$ is arc

$\boxed{x'(t) = 1,\ y'(t) = 2t}$ → $t = x \Rightarrow y = x^2$



Ex:- 2) let

$$z(t) = \begin{cases} 1 - (1-t)i, & 0 \le t \le 1 \\ 1 + t - i, & -1 \le t \le 0 \end{cases}$$

then $z(t)$ is an arc but it is Not a differentiable arc since $z'(0)$ does not exist

$$x(t) = \begin{cases} 1 + t & -1 \le t \le 0 \\ 1 & 0 \le t \le 1 \end{cases}$$

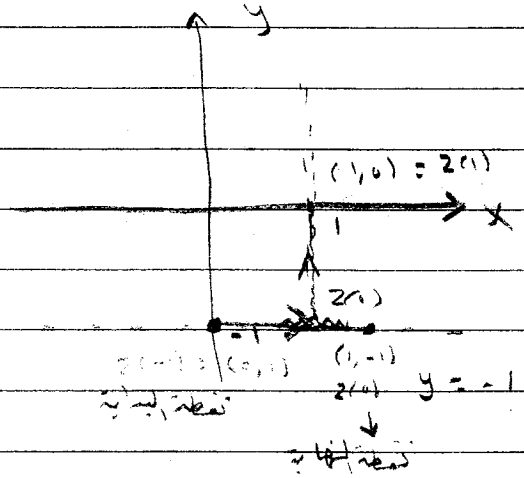
$$y(t) = \begin{cases} -1 & , \quad -1 \le t \le 0 \\ t-1 & , \quad 0 \le t \le 1 \end{cases}$$

$z(t) = x(t) + iy(t)$ ($x(t)$ ، $y(t)$ متصلة اذن هو قابل للاشتقاق + curve)

سوف نرسم هذا المنحنى

In $-1 \le t \le 0$: $x = 1+t$, $y = -1$

" $0 \le t \le 1$: $x = 1$, $y = t-1$



اتجاه الحركة من الصغير الى الكبير
المنحنى غير قابل للاشتقاق عند الصفر

Remark: If $z(t)$, $t \in [a,b]$ is a differentiable arc, then its length L is given by

$$L = \int_a^b |z'(t)|\,dt$$

اشتقاق قانون الطول $\Longrightarrow$

$$\left(\int_a^b \sqrt{1 + [f(x)]^2}\, dx \right)$$

$y = f(x)$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt}$$

$$= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{y'(t)}{x'(t)}\right)^2}\; x'(t)$$

$$= \int_a^b \sqrt{x'^2 + y'^2}\, dt$$

$z' = x' + iy'$

Ex:- Find the length of the arc

$z = e^{i\theta}, \theta \in [0, \pi]$, $z(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$

Solution:- $z'(\theta) = -ie^{i\theta}$ الدالة موجودة

مستمرة وقابلة للاشتقاق

$$L = \int_0^{\pi} |z'(\theta)| d\theta$$

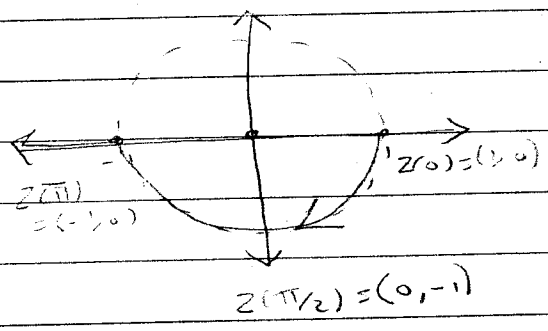
$$= \int_0^{\pi} |-ie^{i\theta}| d\theta$$

$$= \int_0^{\pi} d\theta = \pi$$

نرسم المنحنى وذلك عن طريق كتابة x و y بدلالة θ

$x = \cos\theta$, $y = -\sin\theta$

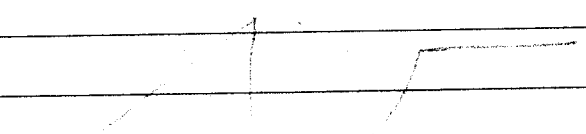
$x^2 + y^2 = 1$ ⟹ circle



Def:- 1) Let C be an arc described by $z = z(t)$, $t \in [a,b]$. If $z'(t)$ exists and cont. for all $t \in [a,b]$ and $z'(t) \neq 0$ for all $t \in (a,b)$, then C is called a smooth arc.

2) A contour, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end.

مثل



smooth arc

مجموعة من المنحنيات الملساء

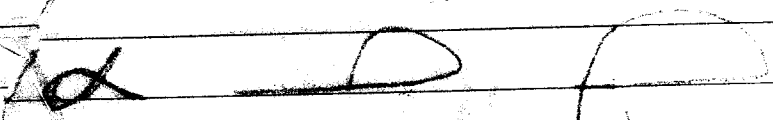
3) Let C be a contour described by

$$z = z(t), \quad t \in [a,b]$$

Then C is called a simple contour if C does not cross itself, that is

if $z(t_1) \neq z(t_2)$ for all $t_1, t_2 \in [a,b]$ with $t_1 \neq t_2$, except that the possibility that $z(a) = z(b)$

يقال انه منحنى بسيط اذا لم يقطع نفسه مثل (يعني المنحنى الذي مر بنفس النقطة)



وايضا المنحنى البسيط في حالة نقطة البداية تساوي نقطة النهاية

4) Let C be a contour described by

$$z = z(t), \quad t \in [a,b]$$

If C is simple except that $z(a) = z(b)$, then C is called a simple closed contour.

Ex:- 1) Consider the unite circle

$$z(t) = e^{it}, \quad t \in [0, 2\pi]$$ (هذا هو دائرة الوحدة)

it is a smooth arc, since $z'(t) = ie^{it}$ is cont. on $[0, 2\pi]$ and $z'(t) \neq 0$ for all $t \in (0, 2\pi)$

it is a contour

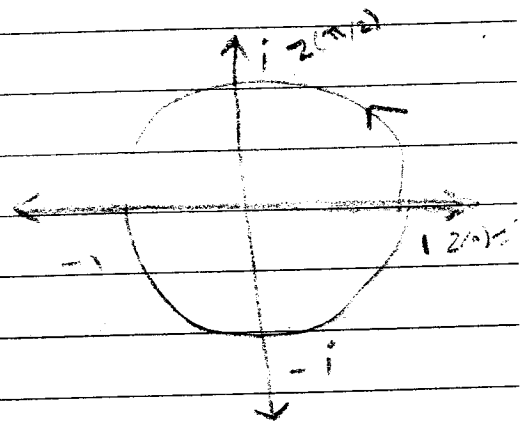
it is a closed simple contour.

Sol: $z(t) = e^{it} = \cos t + i \sin t$, $0 \leq t \leq 2\pi$

t تمثل الزاوية

$z'(t) = ie^{it}$, $z(t) \neq 0$

simple closed contour



Example 2) The unite circle

$$z(t) = e^{it}, \quad t \in [0, 3\pi]$$

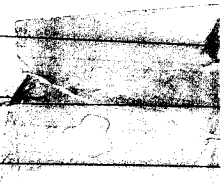
is not a simple curve since

$$z(0) = z(2\pi) = 1$$

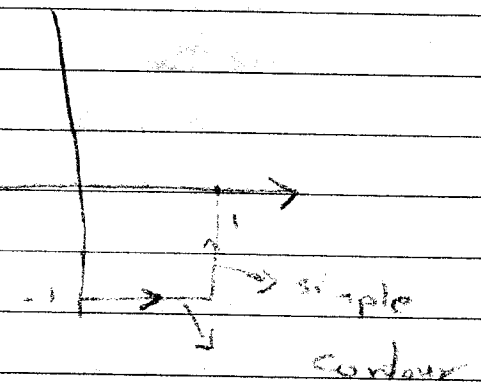
نفس المنحنى ولكن تغير الفترة اعطتنا حالة حيث انه لا يمثل منحنى بسيط لانه يمر مرتين اكثر من مرة.

Example 3) Let

$$z(t) = \begin{cases} 1 - (1-t)i, & 0 \le t \le 1 \\ 1 + t - i, & -1 \le t \le 0 \end{cases}$$



Then $z(t)$ is not a smooth arc. However it is a piecewise smooth arc, that is a simple contour.



Example 4) Consider the arc

$$z(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t + (t-1) & 1 \le t \le 2 \\ (3-t)i & 2 \le t \le 3 \end{cases}$$

Then $z(t)$ is a simple closed contour.

Contour Integrals

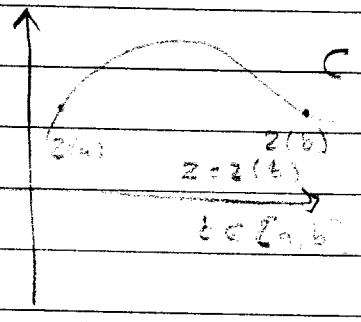
In this section, we turn to integrals of complex-valued functions of a complex variable along a given contour C.

Def. Suppose that the equation

$$z(t) \quad t \in [a,b]$$

represents a contour C.

Let f be a piecewise cont. function on C.



$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$\left(\frac{dz}{dt} = z'(t)\right) \quad \therefore dz = z'(t)\,dt$$

هذا هو تعريف التكامل كما عرفنا

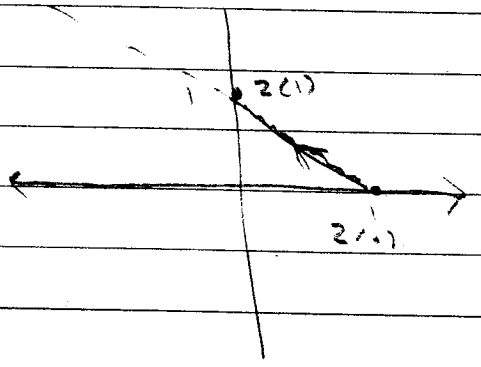
Then we define the line integral, or contour integral, of f along C as:

$$\int_C f(z)\,dz = \int_a^b \underbrace{f(z(t))\,z'(t)}_{w(t)}\,dt$$

Ex. 1) Evalute $\int_C z\,dz$ if C is the contour $z(t) = 1 - t + it$, $t \in [0,1]$

Sol. $\int_C z\,dz$, $f(z) = z$, $z(t) = 1 - t + it$, $0 \le t \le 1$

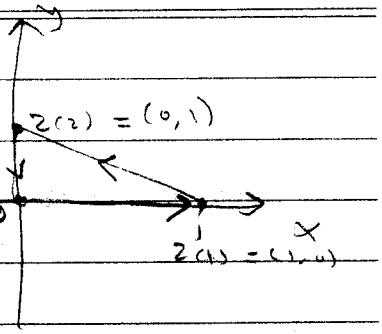
$$\left.\begin{aligned} x &= 1 - t \\ y &= t \end{aligned}\right\} \Rightarrow x = 1 - y \quad y = 1 - x$$



when $x = 1 \Rightarrow y = 1$

$x = 0 \Rightarrow x = 1$

Sol:- $0 \le t \le 1$, $x = t$, $y = 0$



$1 \le t \le 2$, $x = 2 - t$.

$y = t - 1$

$t = y + 1$

$x = 2 - (y + 1)$

$= 1 - y$

$y = 1 - x$

$2 \le t \le 3$, $x = 0$, $y = 3 - t$

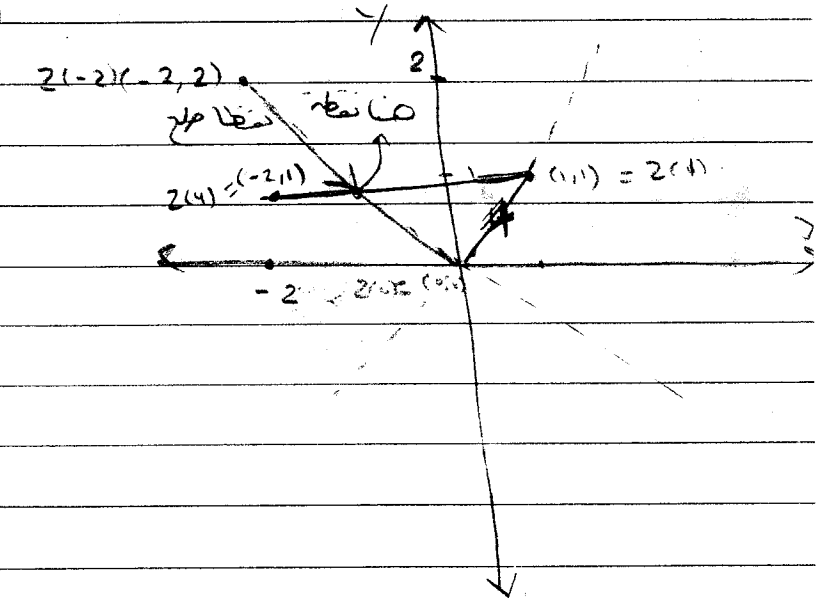
اذن $z(t)$ هو contour لانه هو عبارة عن ثلاث قطع مستقيمة متصلة
و لا يقطع نفسه ونقطة البداية = نقطة النهاية.

Example-5) Let

$$z(t) = \begin{cases} t - it & -2 \le t \le 0 \\ t + it & 0 \le t \le 1 \\ 2 - t + i & 1 \le t \le 4 \end{cases}$$

Then $z(t)$ is not a simple contour since

$z(3) = -1 + i = z(-1)$

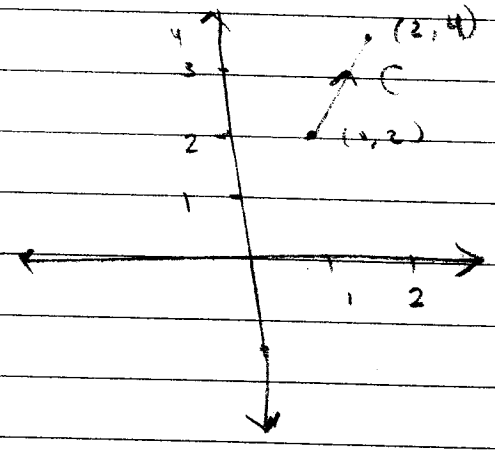


اذن هذا المثال كنتور، غير بسيط
لانه يقطع نفسه.

هذا المثال توضيح للملاحظة (2) السابقة

Ex. Evalute $\int_C \frac{1}{z}\,dz$ if C is the line segment from $(1,2)$ to $(2,4)$

Sol. Use two different representations.



$z = z(t)$

Equation for the line

$$\frac{y-4}{x-2} = \frac{4-2}{2-1} = 2$$

نفرض

$$y - 4 = 2x - 4 \Rightarrow y = 2x \quad , \quad x = t$$

① $\Rightarrow z(t) = t + 2ti \quad , \quad t \in [1,2]$

② $y = s \Rightarrow z(s) = \frac{1}{2}s + si \quad , \quad s \in [2,4]$

سنثبت ان التكامل باستخدام $z(t)$ او $z(s)$ سيعطينا نفس الناتج

$z(t) = t + 2ti \; , \; z'(t) = 1 + 2i$

① $$\int_C \frac{1}{z}\,dz = \int_1^0 \frac{1}{t+2ti}(1+2i)\,dt$$

$$= \int_1^2 \frac{1}{t}\,dt = \ln t \Big|_1^2 = \ln 2$$

② $z = \frac{1}{2}s + si \quad , \quad 2 \le s \le 4$

$$\int_C \frac{1}{z}\,dz = \int_2^4 \frac{1}{\frac{1}{2}s + si}\left(\frac{1}{2} + i\right) ds$$

$$= \int_2^4 \frac{ds}{s} = \ln|s| \Big|_2^4 = \ln 2 .$$

$$\int_C z\,dz = \int_0^1 z(t)\,z'(t)\,dt$$

$$z'(t) = -1+i$$

$$\Rightarrow \int_C z\,dz = \int_0^1 [1-t+it](-1+i)\,dt$$

$$= (-1+i)\int_0^1 (1-t+it)\,dt$$

$$= (-1+i)\left[t - \frac{1}{2}t^2 + i\frac{t^2}{2}\right]_0^1$$

$$= (-1+i)\left[1 - \frac{1}{2} + \frac{1}{2}i\right]$$

$$= (-1+i)\left(\frac{1+i}{2}\right) = -1$$

Remark:

1- Since C is a contour, $z'(t)$ is also piecewise cont. on the interval $[a,b]$, and hence $\int_a^b f(z(t))\,z'(t)\,dt$ exists.

2- The value of the contour integral is interval under a change in the representation of the contour.

Namely, if we use the change of variables $t = T(s)$, where $T(s)$ maps the interval $[A,B]$ into the interval $[a,b]$, $T'(s)$ is cont and $T'(s) > 0$, for all $s \in [A,B]$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_A^B f(z(s))\,z'(s)\,ds.$$

Example: 1) Evalute

$$\int_C (z^2+\bar{z})\,dz$$

and

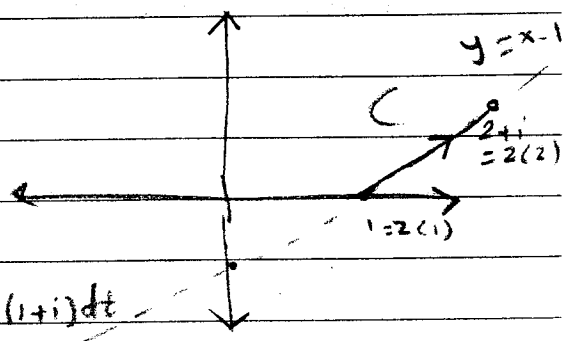
$$\int_{-C} (z^2+\bar{z})\,dz$$

if C is the contour $z(t)=t+i(t-1)$, $t\in[1,2]$.

Sol:- C: $z(t)=t+i(t-1)$

$x=t$, $y=t-1$, $t\in[1,2]$

$y=x-1$



$z'=1+i$

$$\int_C (z^2+\bar{z})\,dz = \int_1^2 [(t+i(t-1))^2+t-i(t-1)](1+i)\,dt$$

$$=(1+i)\int_1^2 [t^2-(t^2-2t+1)+2it(t-1)+t-i(t-1)]\,dt$$

$$=(1+i)\int_1^2 [3t-1+i(2t^2-3t+1)]\,dt$$

$$=(1+i)\left[\frac{3}{2}t^2-t-i\left(\frac{2}{3}t^3-\frac{3}{2}t^2+t\right)\right]_1^2$$

$$=(1+i)\left[\frac{3}{2}(4-1)-(2-1)+i\left(\frac{2}{3}(8-1)-\frac{3}{2}(4-1)+(2-1)\right)\right]$$

$$=(1+i)\left[\frac{9}{2}-1+i\left(\frac{14}{3}-\frac{9}{2}+1\right)\right]$$

$$=(1+i)\left[\frac{7}{2}+i\,\frac{28-27+6}{6}\right)$$

$$=(1+i)\left(\frac{7}{2}+\frac{7i}{6}\right)$$

Proposition :-

1- If z_0 is a complex constant, then

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$$

2- $$\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$$

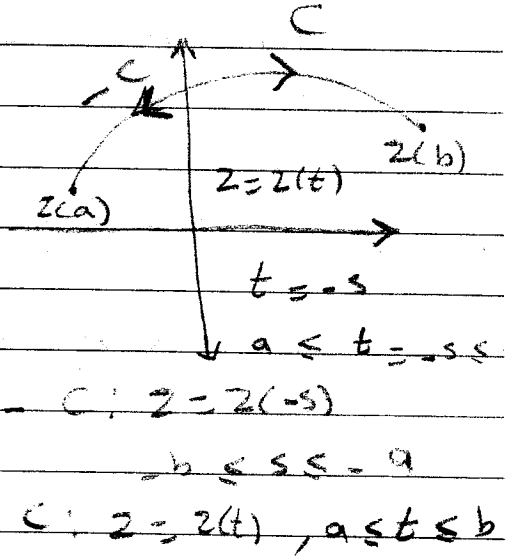
Def:- let C be a contour represented by

$$z = z(t), \quad t \in [a,b]$$

with $z_1 = z(a)$ and $z_2 = z(b)$.

The contour $-C$ consists of the same set of points but with reversed order so that it extends from z_2 to z_1.

The contour $-C$ has parametric representation



Proposition:-

1- Let C be a contour consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3 Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

2- If $|f(z)| \le M$ for all $z \in C$ and L is the length of C, then

$$\left| \int_C f(z)\,dz \right| \le ML.$$

Proof:- 2)

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\,z'(t)\,dt \right|$$

$$\le \int_a^b |f(z(t))\,z'(t)|\,dt$$

$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\le M \int_a^b |z'(t)|\,dt = ML.$$

Examples:- In this section, we will see examples to illustrate how to use the definition to evalute contour integrals and the properties of these integrals.

19

$-C : z(-s) = -s + i(-s+1)$

$= -s - i(s+1) \qquad -2 \le s \le -1$

$x = -s \;, \quad y = -s - 1 \Rightarrow y = x - 1$

$$\int_{-C} (z^2 + \bar{z})\, dz = \int_{-2}^{-1} \left([-s - i(s+1)]^2 - s + i(s+1) \right)(-1 - i)\, ds$$

$$= -(1+i) \int_{-2}^{-1} [\;\;]\, ds = -(1+i)\left(\frac{x}{2} + \frac{x}{6} i\right)$$

Proposition:- $\int_{-\epsilon} f(z)\, dz = -\int_{\epsilon} f(z)\, dz$

proof:- $C : z(t), \quad a \le t \le b$

$-C : z(-s), \quad -b \le s \le -a \;, \quad \frac{dz}{ds} = -z'(-s)\, ds$

$$\int_{-C} f(z)\, dz = -\int_{-b}^{-a} f(z(-s))\, z'(-s)\, ds$$

$t = -s \Rightarrow dt = -ds$

$$\therefore \int_{-C} f(z)\, dz = -\int_{t=b}^{t=a} f(z(t))\, z'(t)\, (-dt)$$

$$= -\int_{a}^{b} f(z(t))\, z'(t)\, dt$$

$$= -\int_{\epsilon} f(z)\, dz .$$

Example 1):- Evalute $\int_C Re(z)\,dz$ if C is the circle $|z|=2$

عكس عقارب الساعة

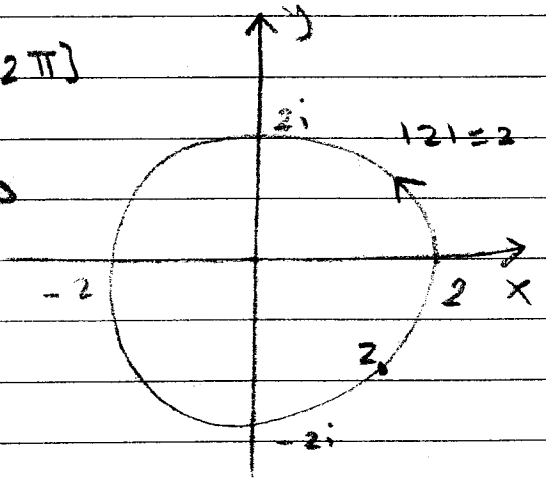
discribed in the positive (counterclockwise) direction.

sol:- $f(z)=Re(z)$, $C: |z|=2$

$z=|z|e^{i\arg z}$

$=2e^{it}$ ($\arg z$ متغير اذا أمكن اعتبرها t) $t\in[0,2\pi]$

هذه طريقة عمل parameterization للدائرة



$z'=2ie^{it}$

$$\int_C Re\,z\,dz=\int_0^{2\pi} Re(2e^{it})\,2ie^{it}\,dt$$

$$=2i\int_0^{2\pi} 2\cos t\,[\cos t+i\sin t]\,dt$$

$$=4i\int_0^{2\pi}(\cos^2 t+i\sin^2 t\cos t)\,dt$$

$$=4i\left[\int_0^{2\pi}\frac{1+\cos 2t}{2}+\underbrace{\int_{u=0}^{u=0} u\,du}_{=0}\right]$$

$u=\sin t$

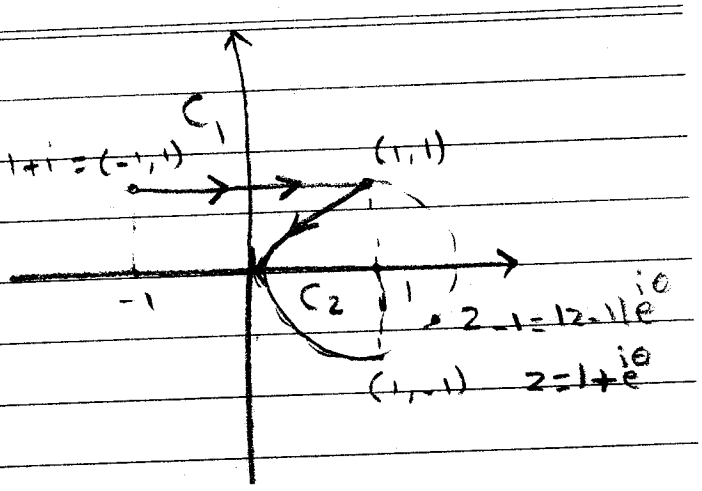
$du=\cos t\,dt$

$$=2i\left[t+\frac{1}{2}\sin 2t\right|_0^{2\pi}$$

$$=2i\,[2\pi]=4\pi i$$

Example:- 2) Evalute $\int_C (z^2+1)\,dz$ if C consists of the line seg from $(-1,1)$ to $(1,1)$ followed by the left-hand half of circle $|z-1|=1$.

Sol:- $C = C_1 + C_2$



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

C_1: $y=1 \Rightarrow z = t+i \;,\; t\in[-1,1]$

$$z' = 1$$

$$\int_{C_1} (z^2+1)\,dz = \int_{-1}^{1} [(t+i)^2+1]\,dt$$

$$= \int_{-1}^{1} [t^2+2ti]\,dt$$

$$= \frac{1}{3}t^3 + it^2 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(1+1) = \frac{2}{3}.$$

C_2: $z = 1 + e^{i\theta} \;,\; \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

المركز الدائرة

نصف الدائرة

$$z' = ie^{i\theta}$$

$$\int_{C_2} (z^2+1)\,dz = \int_{\pi/2}^{3\pi/2} [(1+e^{i\theta})^2+1]\, ie^{i\theta}\,d\theta$$

$$= i\int_{\pi/2}^{3\pi/2} [2+2e^{i\theta}+e^{2i\theta}]\,e^{i\theta}\,d\theta$$

$$= i\left[\frac{2e^{i\theta}}{i} + \frac{2e^{2i\theta}}{2i} + \frac{e^{3i\theta}}{3i}\right]_{\pi/2}^{3\pi/2}$$

$$= 2\left[e^{i\,3\pi/2} - e^{i\,\pi/2}\right] + \left[e^{3\pi i} - e^{\pi i}\right] + \frac{1}{3}\left[e^{9\pi/2\, i} - e^{3\pi/2\, i}\right]$$

$$= 2\left[-i - i\right] + \left[-1 - (-1)\right] + \frac{1}{3}\left[i - (-i)\right]$$

$$= -4i + \frac{2i}{3} = \frac{-10i}{3}$$

$$\int_C (z^2+1)\,dz = \frac{2}{3} - \frac{10i}{3}.$$

Example: 3) Let $f(z) = z^{1/2}$ be the principle branch. Evalute $\int_C z^{1/2}$ if C is the semicircular path $z = 4e^{i\theta}$, $\theta \in [0,\pi]$.

Sol: $f(z) = \sqrt{|z|}\, e^{i/2\, \text{Arg}\, z}$, $-\pi < \text{Arg}\, z < \pi$

$f(z) = \sqrt{|z|}\, e^{1/2\, i\, \text{Arg}\, z} = 2e^{i/2\,\theta}$

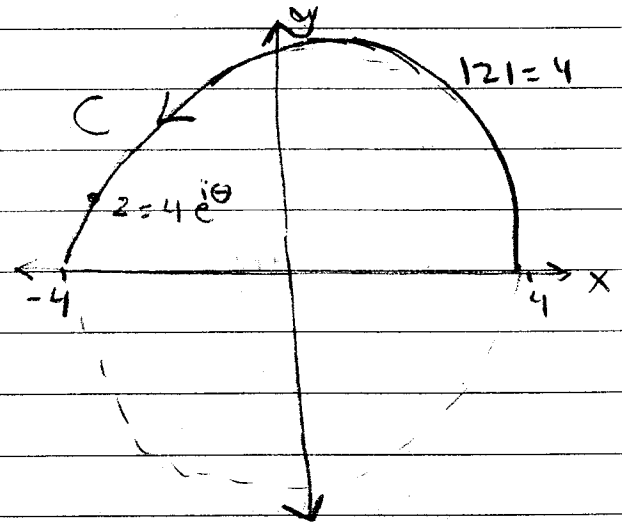
$\theta \in [0,\pi] \Rightarrow \theta = \text{Arg}\, z$

$z' = 4i\, e^{i\theta}$, $C: z = 4e^{i\theta}$.

$$\int_C f(z)\,dz = \int_0^{\pi} 2e^{i/2\,\theta}\, 4i\, e^{i\theta}\, d\theta$$

$$= 8i \int_0^{\pi} e^{3/2\, i\theta}\, d\theta = 8i\, \frac{e^{3/2\, i\theta}}{\frac{3}{2} i}$$

$$= \frac{16}{3}\left[e^{3\pi/2 i} - e^{0}\right] = \frac{16}{3}(-i - 1)$$



Example 4) Show that $\left|\int_C e^z dz\right| \leq 2\pi e$, where C is the unit circle $|z|=1$.

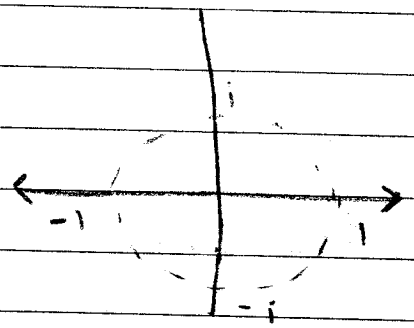
Sol:- $\left|\int_C f(z)dz\right| \leq ML$

L = the length of C, $M = \max_{z\in C}|f(z)|$ لم يحطي اتجاه لانه دائرة

طول الدائرة يعني محيط الدائرة:

$L = 2\pi \times r = 2\pi \times 1 = 2\pi$

$|f(z)| = |e^z| = e^x \leq e \quad \forall z \in \mathbb{C}$



$M = e$

$\left|\int_C e^z dz\right| \leq 2\pi e.$

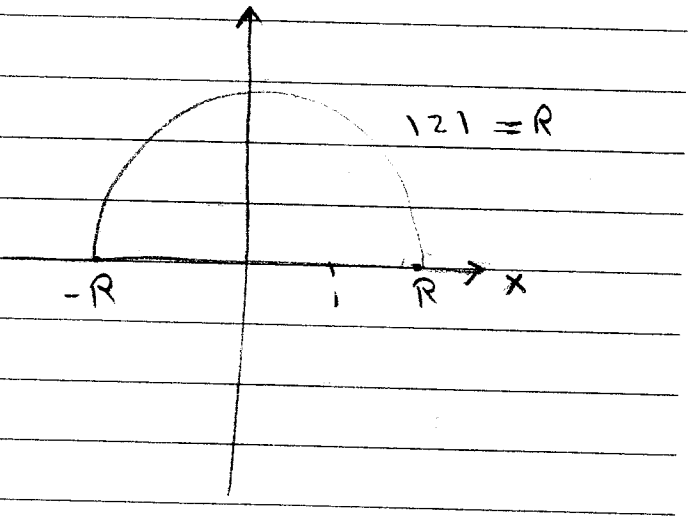
Example 5) Show that $\left|\int_C \frac{1}{z^2-1} dz\right| \leq \frac{\pi R}{R^2-1}$ where C is the upper semicircle.

Sol:- $\left|\int_C f(z)dz\right| \leq LM$

L = the length of C

$= \pi R$ (محيط نصف دائرة)

$M = \max_{z\in C}|f(z)|, \quad |f(z)| = \left|\frac{1}{z^2-1}\right|$



$|z^2-1| \geq ||z|^2-1| = |R^2-1| = R^2-1$

$|f(z)| = \frac{1}{|z^2-1|} \leq \frac{1}{R^2-1} = M$

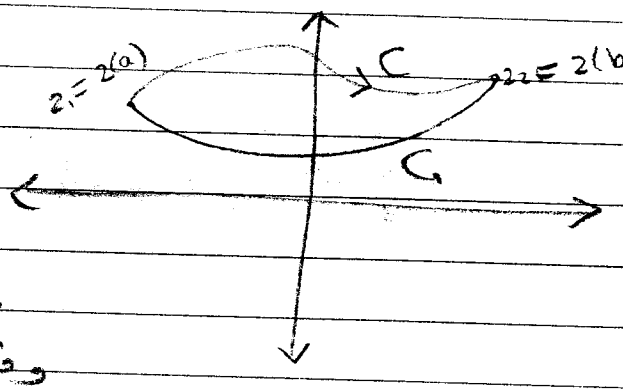
$\therefore \left|\int_C f(z)\,dz\right| \leq LM = \frac{\pi R}{R^2-1}.$

Antiderivatives

In general the contour integral of a function $f(z)$ from a point z_1 to a point z_2 depends on the path that is taken. However there are functions whose integrals from z_1 to z_2 are independent of path. For such functions we will show that

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1). \qquad (*)$$

where F is aniderivative of f and C is any contour from z_1 to z_2.



المعادلة (*) هي مشتقة الدالة $f(z)$
وواضح من التعريف ان التكامل يعتمد
على نقطة البداية والنهاية فقط ولا يعتمد
على مسار المنحني وهذا المفهوم خطأ
وفيما يلي مثال يوضح ذلك حيث ان مسار
المنحني مهم في ايجاد التكامل

Ex:- 1) Find $\int_{C_1} |z|^2\,dz$ and $\int_{C_2} |z|^2\,dz$ if

$$C_1 : z_1(t) = t + it \ , \ t \in [0,1]$$

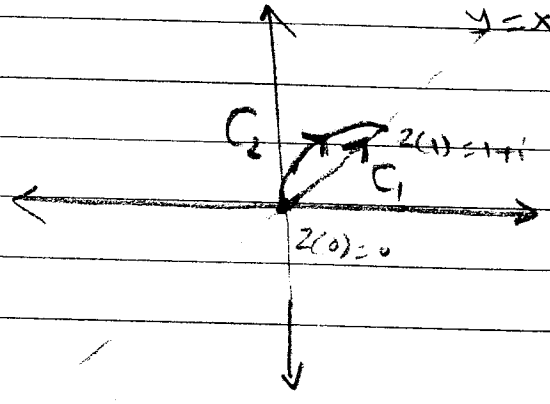
and

$$C_2 : z_2(t) = t^2 + it \ , \ t \in [0,1]$$

sol:- $C_2 : x = t^2 \ , y = t \Rightarrow x = y^2$

$C_1 : z = t + it \ , \ 0 \le t \le 1$

$z' = 1 + i$



$$\int_{C_1} |z|^2\,dz = \int_0^1 |t + it|^2 (1+i)\,dt$$

$$= (1+i)\int_0^1 2t^2\,dt$$

$$= (1+i)\,\frac{2}{3}t^3\Big|_0^1 = \frac{2}{3}(1+i)$$

C_2: $z = t^2 + it$, $0 \le t \le 1$

$z' = 2t + i$

$$\int_{C_2} |z|^2\,dz = \int_0^1 |t^2+it|^2(2t+i)\,dt$$

$$= \int_0^1 (t^4+t^2)(2t+i)\,dt$$

$$= \int_0^1 [2t^5 + 2t^3 + i(t^4+t^2)]\,dt$$

$$= \frac{1}{3}t^6 + \frac{1}{5}t^4 + i\left(\frac{1}{5}t^5 + \frac{1}{3}t^3\right)\Big|_0^1$$

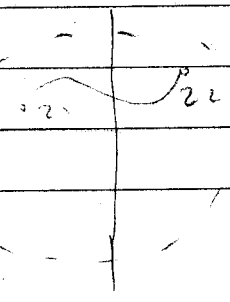
$$= \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{5}+\frac{1}{3}\right) = \frac{5}{6} + \frac{8}{15}i$$

$$\therefore \int_{C_1} |z'|\,dz + \int_{C_2} |z|^2\,dz$$

Proposition (Fundamental theorem for complex integration):

Suppose that a function f is continuous on a domain D and f has an antiderivative F in D, that is, there is a function F such that $F'(z) = f(z)$ for all $z \in D$. Then for any contour C lying in D with end points z_1 and z_2

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$



In particular, if the counter C is closed then

$$\int_C f(z)\,dz = 0.$$

النظرية السابقة تعطي الشروط التي يكون فيها التكامل يعتمد فقط على النهايات

Proof:- $$\int_C f(z)\,dz = \int_C F'(z)\,dz = \int_a^b F'(z(t))\,z'(t)\,dt$$

$$\frac{d}{dt} F(z(t)) = F'(z(t))\,z'(t)$$

$$\Rightarrow \int_C f(z)\,dz = \int_a^b \frac{d}{dt}\left[F(z(t))\right] dt$$

$$= F(z(t))\Big|_a^b = F(z(b)) - F(z(a))$$

$$= F(z_2) - F(z_1)$$

Remark:- If the function $f(z)$ satisfies the hypothesis of the proposition, then for all contours C lying in D begining at z_1 and ending at z_2 we have

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

Hence the result demonstrates that the integral is independent of path.

Examples:- Let us give some examples to illustrate the above proposition

Example 1):- Use an antiderivative to evalute

$$\int_C z\,dz$$

if C is the contour $z(t) = 1 - t + it$, $t \in [0,1]$.

Sol:- $f(z) = z \Rightarrow F(z) = \frac{1}{2}z^2$, $f(z)$ is cont. on $\mathbb{C}$

$$\therefore \int_C z\,dz = F(z)\Big|_{z_1 = z(0) = 1}^{z_2 = z(1) = i}$$

$$= \frac{1}{2}z^2\Big|_1^i = \frac{1}{2}(i^2 - 1) = -1$$

Example 2) Use an antiderivative to evalute

$$\int_C \frac{1}{z}\,dz$$

if C is the contour $z(t) = e^{it}$, $t \in [0, 5\pi/4]$.

دائرة: نصف قطرها = 1، مركزها 0

Sol:- $f(z) = \frac{1}{z} \Rightarrow F(z) = \log z = \ln|z| + i\arg z$

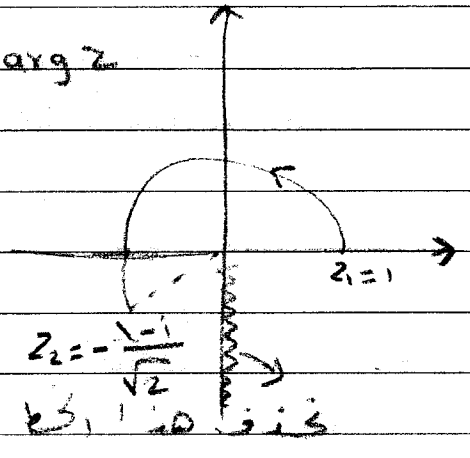
$$-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

$$\int_C \frac{1}{z}\,dz \log z\Big|_1^{\frac{-1-i}{\sqrt{2}}}$$

$$= \log\left(\frac{-1-i}{\sqrt{2}}\right) - \log i$$

$$= \ln 1 + i\frac{5\pi}{4} - [\ln 1 + i\,0]$$

$$= i\frac{5\pi}{4}.$$

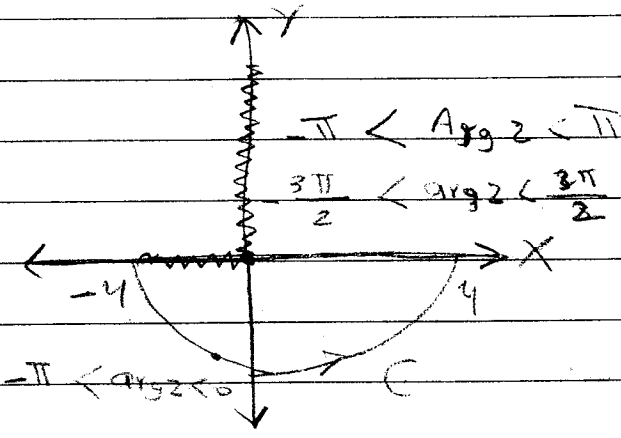


$$\left[\operatorname{Arg}\frac{-1-i}{\sqrt{2}} = -\pi + \tan^{-1}\left(\frac{y}{x}\right) = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}\right]$$

Example 3) Use an antiderivative to evalute

$$\int_C z^{1/2}\, dz,$$

where $z^{1/2} = \sqrt{|z|}\, e^{i \operatorname{Arg} z}$, $-\pi < \operatorname{Arg} z < \pi$ and C is the contour $z(t) = 4e^{it}$, $t \in [-\pi, 2\pi]$



Sol. $f(z) = z^{1/2} = \sqrt{|z|}\, e^{i \operatorname{Arg} z}$

$f_1(z) = z^{1/2} = \sqrt{|z|}\, e^{i/2 \arg z}$

$F_1(z) = \dfrac{z^{3/2}}{3/2}$

$F_1(z) = f(z) \quad \forall z \in C, \forall z \in \mathbb{C} - (-\infty, -4]$

$$\int_C f(z)\, dz = \int_C f_1(z)\, dz = \frac{2}{3} z^{3/2} \Big|_{-4}^{4}$$

$$= \frac{2}{3}\left[(64)^{1/2} - (64)^{1/2}\right] = \frac{2}{3}\left[8 e^{\frac{1}{2} \cdot 0} - 8 e^{-\frac{i}{2}\pi}\right]$$

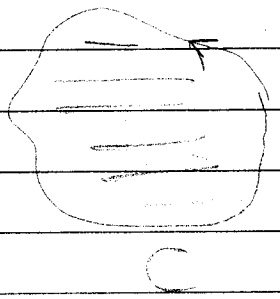
$$= \frac{16}{3}(1 + i).$$

Theorem: (Cauchy-Goursat):

Let C be a simple closed contour and let $f(z)$ be a function that is analytic in the interior of C as well as on C itself. Then

$$\int_C f(z)\,dz = 0.$$

proof: The proof depends on the Green's theorem.

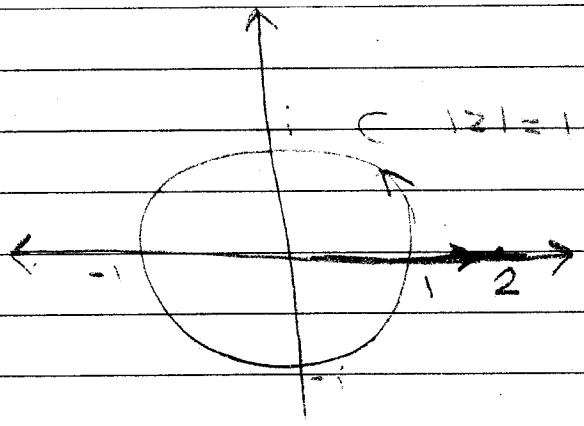


C

Example 1): Show that

$$\int_C \frac{z}{z-2}\,dz = 0$$

if C is the unit circle $|z|=1$



Sol: $f(z) = \frac{z}{z-2}$

$f(z)$ is analytic on C and [illegible] enclosed by C

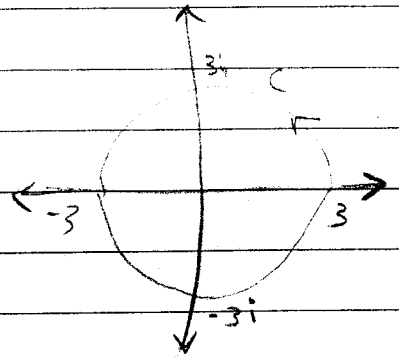
$$\Rightarrow \int_C f(z) = 0$$

Example 2): Show that

$$\int_C \frac{1}{z}\,dz = 2\pi i$$

if C is the positively oriented circle $|z|=3$

sol:- $f(z) = \frac{1}{z}$



is not analytic at $z=0$

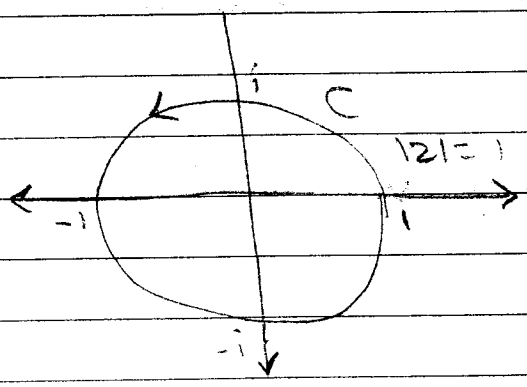
$z=0 \in$ the region enclosed by C

$C: |z|=3 \Rightarrow z = 3e^{it},\ t \in [0, 2\pi]$

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{3e^{it}}\, 3ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Example:-3) show that $\int_C \frac{1}{z^2}\,dz = 0$ if C is the positively oriented circle $|z|=1$

sol:- $f(z) = \frac{1}{z^2}$



is not analytic at $z=0$, the region enclosed by C.

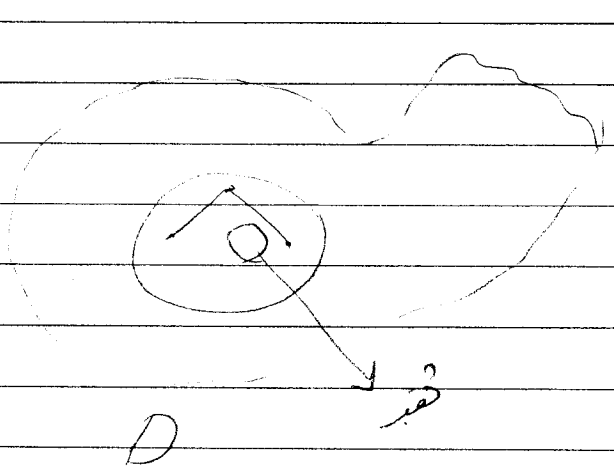
$F(z) = -\frac{1}{z}$, satisfied $F'(z) = f(z)$

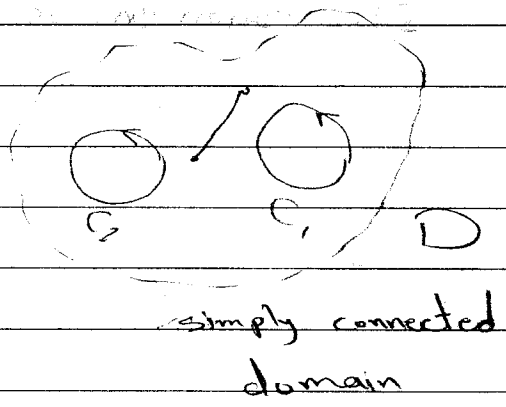
$$\Rightarrow \int_C \frac{1}{z^2}\,dz = 0.$$

Simply and multiple connected domains

In this section, we will give extensions of Cauchy-Goursat theorem

Def: A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D.



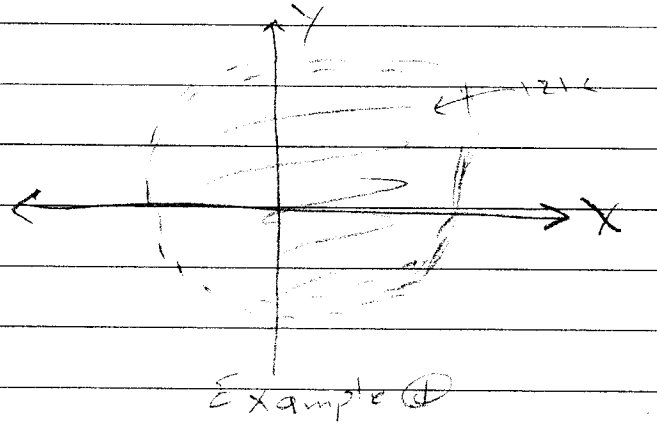


simply connected domain

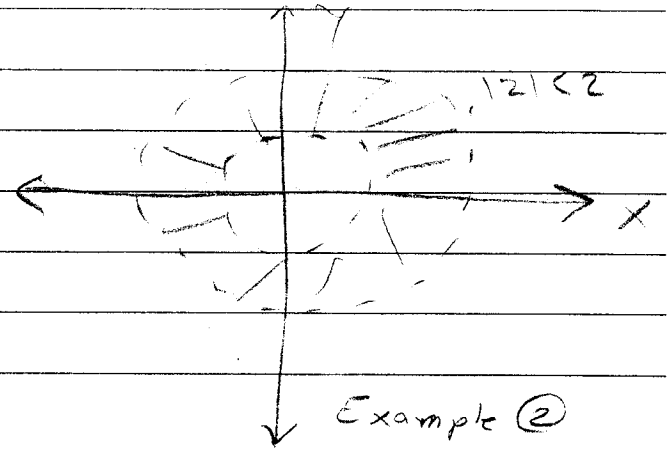
If a domain D is not simply connected, then it is called multiply connected.

Example 1) $D = \{z : |z| < 2\}$ is a simply connected domain

Example 2) $D = \{z : |z| < 2\}$ is a multiply connected domain.



Example ①



Example ②

Theorem: If a function f is analytic throughout a simply connected domain D, then

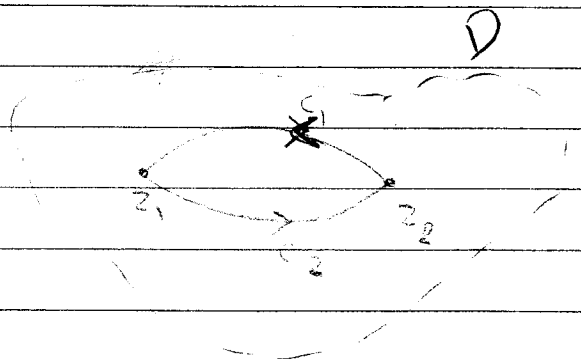
$$\int_C f(z)\,dz = 0$$

for every closed contour C lying in D.

Corollary 1) A function f which is analytic throughout a simply connected domain D must have an antiderivative in D.

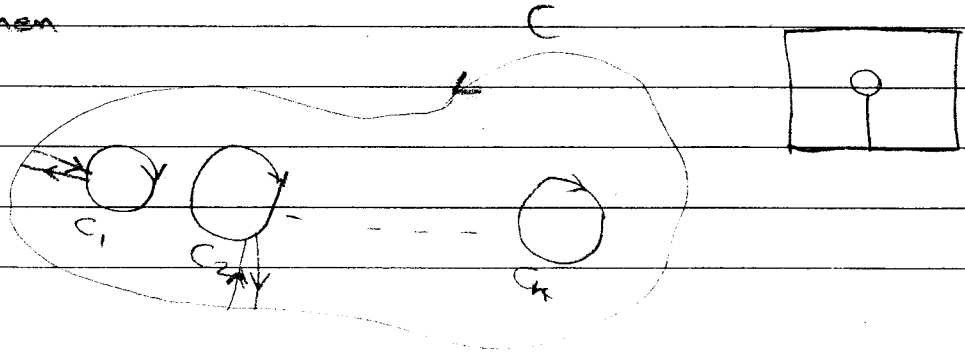
$C_1 - C_2$

$\int_{C_1 - C_2} f(z)\,dz = 0$



$$\int_{C_1} + \int_{-C_2} = 0$$

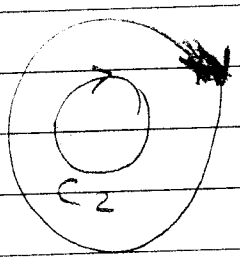
Theorem: Let C be a simple closed contour described in the counterclockwise direction; and let C_k, $k = 1, 2, \ldots, n$, be a simple closed contours described in the clockwise direction, that are interior to C and whose interiors have no points in common.
If f is an analytic function throughout the closed region consisting of all points within and on C except for points interior to any C_k, then



$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

Corollary 2: Let C_1 and C_2 be positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed region consisting of these contours and all points between them, then

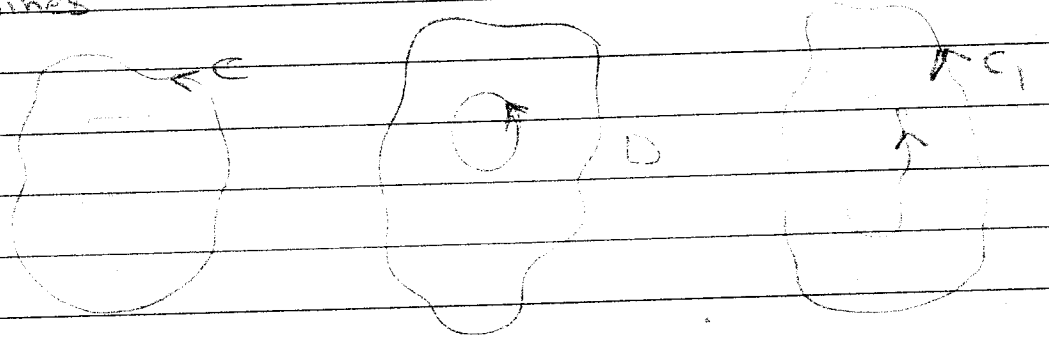
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



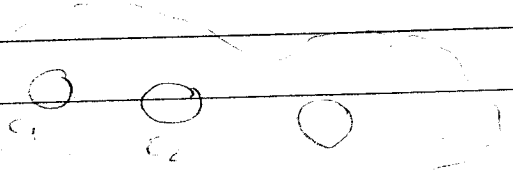
$$\int_{C_1} + \int_{C_2} = 0$$

$$\int_{C_1} = \int_{-C_2} = 0$$

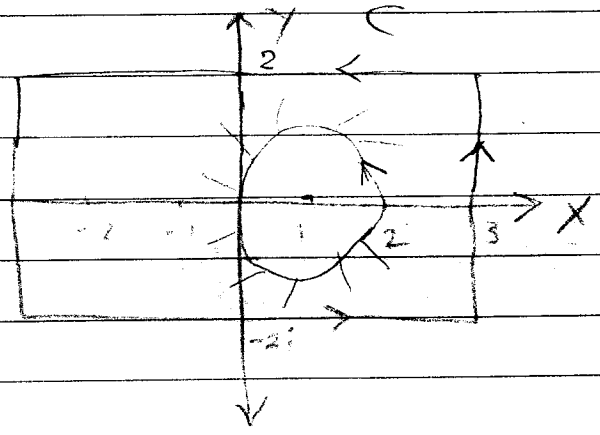
Example 1): Evalute $\int_C \frac{z^3}{(z-1)}\,dz$, where C is the positively oriented boundary of the rectangle whose sides lie along the lines



$$\int_C f(z)\,dz = 0 \quad , \quad \int + \sum \int = 0$$



Example 1) Evalute $\int_C \frac{z^3}{(z-1)^2}\,dz$, where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x=\pm 3$, $y=\mp 2$.



solution: $f(z)=\frac{z^3}{(z-1)^2}$

$C_1: |z-1|=1,\ z=1+e^{it},\ t\in[0,2\pi]$

$f(z)$ is analytic on C and C_1 and the region between them

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz = \int_0^{2\pi} \frac{(1+e^{it})^3}{e^{2it}}\, ie^{it}\,dt$$

$$= i\int_0^{2\pi} e^{-it}\left[1+3e^{it}+3e^{2it}+e^{3it}\right]dt$$

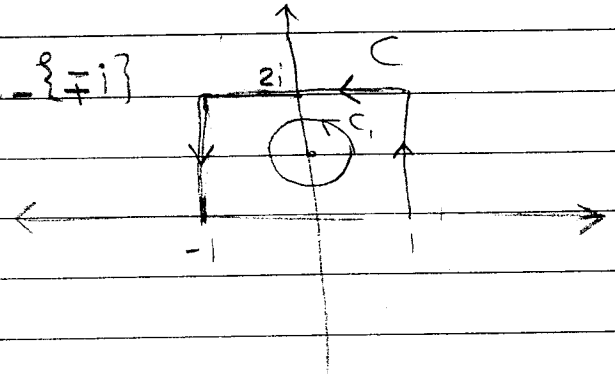
$$= i\left[\frac{e^{-it}}{i}+3t+3\frac{e^{it}}{i}+\frac{e^{3it}}{3i}\right]_0^{2\pi}$$

$$= -\left[e^{-3\pi i}-e^{0}\right]+3i\left[3\pi-0\right]+3\left[e^{2\pi i}-e^{0}\right]+\frac{1}{3}\left[e^{6\pi i}-e^{0}\right]$$

$$= 6\pi i.$$

Example 2) Find $\int_C \frac{2z}{z^2+1}\,dz$ if C is the boundary of the square $-1\le x\le 1$, $0\le y\le 2$ described in the positive sense.

sol: $f(z)=\frac{2z}{z^2+1}$ is analytic on $\mathbb{C}-\{\mp i\}$



$C_1: |z-i|=\frac{1}{2}$

$z=i+\frac{1}{2}e^{it}$, $t\in[0,2\pi]$

f is analytic on C and C_1 and in the region between them

$$\Rightarrow \int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

$$z^2+1 = (z-i)(z+i)$$

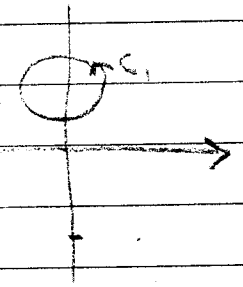
$$\frac{2z}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i}$$

$$2z = A(z+i) + B(z-i)$$

$$z=i \Rightarrow 2i = 2iA \Rightarrow A=1$$

$$z=-i \Rightarrow -2i = -2iB \Rightarrow B=1$$

$$\Rightarrow f(z) = \frac{2z}{z^2+1} = \frac{1}{z-i} + \frac{1}{z+i}$$



$$\Rightarrow \int_C f(z)\,dz = \int_{C_1} \frac{1}{z-i}\,dz + \int_{C_1} \frac{1}{z+i}\,dz$$

$\frac{1}{z+i}$ is analytic on C_1 and in the region enclosed by C_1

$$\Rightarrow \int_{C_1} \frac{1}{z+i}\,dz = 0$$

$$\Rightarrow \int_C f(z)\,dz = \int_0^{2\pi} \frac{1}{\frac{1}{2}e^{it}} \frac{1}{2} i e^{it}\,dt$$

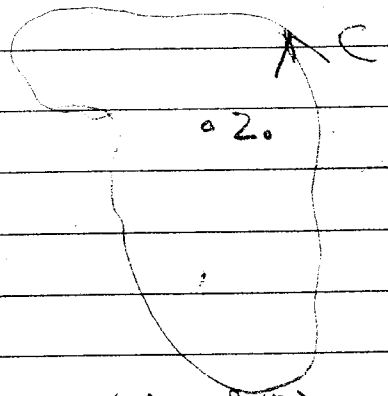
$$= \int_0^{2\pi} i\,dt = it \Big|_0^{2\pi} = 2\pi i$$

Cauchy integral formula

In this section, we discuss a number of fundamental consequences of the results represented in the previous sections.

Theorem: Let f be a function that is analytic everywhere within and on a simple closed contour C taken in the positive sense. Then for any point z_0 interior to C

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz =$$



$$g(z) = \frac{f(z)}{z-z_0}$$

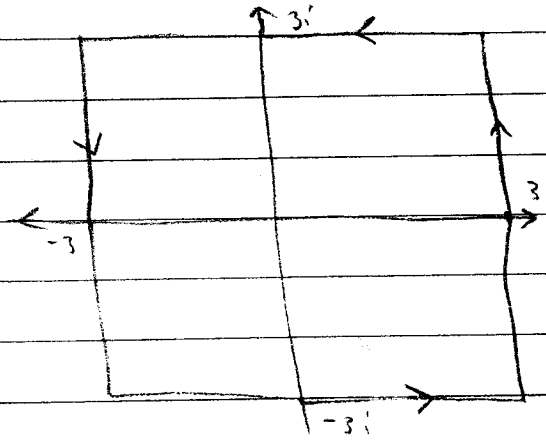
$$\int g(z)\,dz = 2\pi i f(z_0)$$

Example 1) Evalute $\int_C \frac{\sin z}{z+1}\,dz$, where C is the positively oriented boundary of the square $-3 \le x \le 3$, $-3 \le y \le 3$.

sol: $g(z) = \frac{\sin z}{z+1}$ is analytic on $\mathbb{C} \setminus \{-1\}$ and is lies inside C

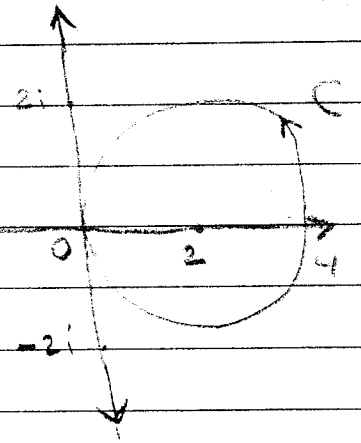
$f(z) = \sin z$ is analytic within and on C, $z_0 = -1$

By Cauchy integral formula



$$\int_C \frac{\sin z}{z+1}\,dz = 2\pi i\, f(z) = 2\pi i \sin(-1)$$
$$= -2\pi i \sin 1$$

Example 2) Evalute $\int_C \frac{z\,dz}{(z^2+4)(z-1)}$, where C the positively oriented circle $|z-2|=2$.



solution $g(z) = \frac{z}{(z^2+4)(z-1)}$

is analytic $\forall z \in \mathbb{C} - \{1, \pm 2i\}$

$$g(z) = \frac{\frac{z}{z^2+4}}{z-1}, \quad f(z) = \frac{z}{z^2+4}$$

$f(z)$ is analytic within and on C

$z_0 = 1$

by C.I formula

$$\int_C \frac{z}{(z^2+4)(z-1)}\,dz = 2\pi i\, f(z) = \frac{2\pi i}{5}$$

Derivatives of analytic functions

In this section, we will proof an important result about analytic functions. This result states that if a function is analytic at a point then its derivatives of all order exist at that point and they are analytic at that point.

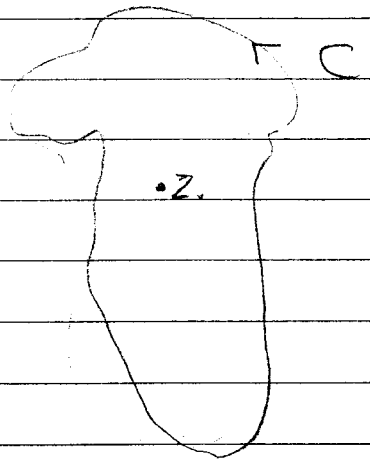
Theorem: Let f be a function that is analytic everywhere within and on a simple closed contour C taken in the positive sense. Then for any point z_0 interior to C all the derivatives

$$f^{(n)}(z_0), \quad n = 0, 1, 2, \ldots$$

exist and

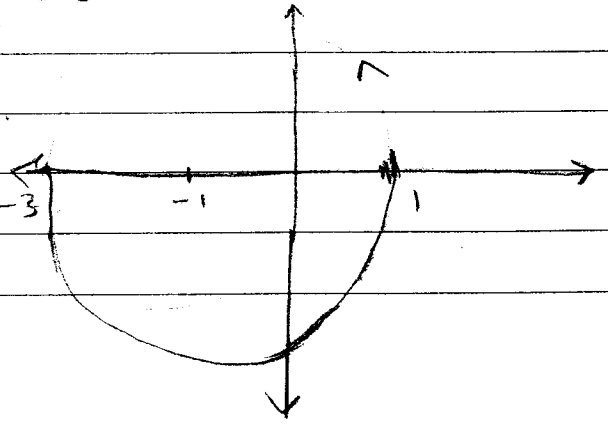
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 0, 1, 2, \ldots$$

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$



Example: Evaluate $\int_C \frac{\cos \pi z}{(2z+1)^3 (z-2)^2} dz$ where C is the positively oriented circle $|z+1| = 2$.

Sol. $g(z) = \frac{\cos(\pi z)}{(2z+1)^3 (z-2)^2}$



g is analytic $\forall z \in \mathbb{C} \setminus \{-\frac{1}{2}, 2\}$

$-\frac{1}{2}$ is inside C and 2 outside C

$$g(z) = \frac{\frac{\cos(\pi z)}{4(z-2)^2} \times f(z)}{(z+\frac{1}{2})^2}, \quad z_0 = -\frac{1}{2}, \quad n = 1$$

$$\int_C \frac{f(z)}{(z+\frac{1}{2})^3} dz = \frac{2\pi i}{1!} f'(-\frac{1}{2})$$

$$f(z) = \frac{1}{4}\cos(\pi z)(z-2)^2$$

$$f'(z) = -\frac{\pi}{4}\sin(\pi z)(z-2)^2 + \frac{1}{2}\cos(\pi z)(z-2)^3$$

$$f'(\tfrac{1}{2}) = \frac{-\pi}{4}\sin(-\tfrac{\pi}{2})(-\tfrac{5}{2})^2 - \frac{1}{2}\cos(-\tfrac{\pi}{2})(-\tfrac{5}{2})^3$$

$$= \frac{\pi}{4}\cdot\frac{1}{\frac{25}{4}} = \frac{\pi}{25}$$

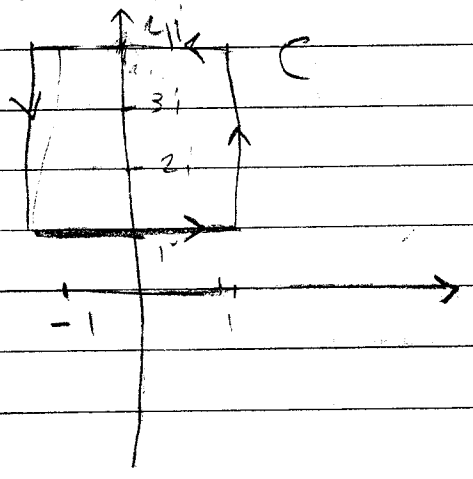
$$\Rightarrow \int_C \frac{\cos(\pi z)}{(2z+1)^2(z-2)^2} dz = 2\pi i \frac{\pi}{25} = \frac{2\pi^2 i}{25}$$

Example2) Evalute $\int_C \frac{z^3+2z+1}{z(z-2i)^3} dz$, where C is the positively oriented rectangle $-1 \le x \le 1, 1 \le y \le 4$.

Solution:- $g(z) = \frac{z^3+2z+1}{z(z-2i)^3}$

is analytic $\forall z \in \mathbb{C} - \{0, 2i\}$

0 is outside C and $2i$ is inside C



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$$g(z) = \frac{\frac{z^3+2z+1}{z}}{(z-2i)^3} \quad \text{, } f(z)$$

$z_0 = 2i$, $n = 2$

$f(z) = z^2 + 2 + \frac{1}{z}$ is analytic on $\mathbb{C}$ and in the region enclosed by $\mathbb{C}$

$$\int_{\mathbb{C}} \frac{f(z)}{(z-2i)^3}\,dz = \frac{2\pi i}{2!} f''(z_0) = \frac{2\pi i}{2!} f''(2i)$$

$f'(z) = 2z - \frac{1}{z^2} \Rightarrow f''(z) = 2 + \frac{2}{z^3}$

$$f''(2i) = 2 + \frac{2}{(2i)^3} = 2 + \frac{2}{-8i} = 2 + \frac{1}{4}i$$

$$\therefore \int_{\mathbb{C}} \frac{f(z)}{(z-2i)^3}\,dz = \frac{2\pi i}{2!} f''(2i) = \pi i \left(2 + \frac{1}{4}\right) = 2\pi i \cdot \frac{\pi}{4}.$$

Corollary 1): If a function f is analytic at a point, then its derivatives of all orders are also analytic functions at the point.

Chapter 5
Series

In this chapter, we study how to use series to represent analytic functions. The best known series are Taylor and Laurent series, which represent analytic functions in appropriate domains. In addition we will study singular points of functions of a complex variable in more details.

Convergence of sequences and series

Definition:-

(1) An infinite sequence of complex numbers is a function from $\mathbb{N}$ to $\mathbb{C}$, $\{z_n\} = z_1, z_2, \ldots$

(2) Let $\{z_n\}$ be a sequence of complex numbers. We say that $\{z_n\}$ has a limit z if, for each $\epsilon > 0$, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon, \quad \forall n > n_0.$$

(3) If a sequence of complex numbers $\{z_n\}$ has a limit z, then we say that the sequence converges to z, and we write

$$\lim_{n \to \infty} z_n = z.$$

(4) If a sequence of complex numbers $\{z_n\}$ does not have a limit, then we say that it diverges.

Theorem: Let $\{z_n\}$ be a sequence of complex numbers and let z be a complex number such that $z_n = x_n + iy_n$ and $z = x + iy$. Then

$$\lim_{n\to\infty} z_n = z = x + iy$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

proof: ($\Leftarrow$) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

$$\Rightarrow \lim_{n\to\infty} z_n = z = x + iy$$

Assume $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Let $\epsilon > 0$ be given we have to find a positive integer n_0 such that $|z_n - z| < \epsilon \ \forall n > n_0$.

$\lim_{n\to\infty} x_n = x \Rightarrow \exists n_1 > 0$ such that

$$|x_n - x| < \frac{\epsilon}{2} \quad \forall n > n_1$$

$\lim_{n\to\infty} y_n = y \Rightarrow \exists n_2 > 0$ such that

$$|y_n - y| < \frac{\epsilon}{2} \quad \forall n > n_2$$

Let $n_0 = \max\{n_1, n_2\}$

Then $|z_n - z| = |x_n - x + i(y_n - y)|$

$$\leq |x_n - x| + |y_n - y|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n > n_0 \qquad \square$$

Proof: $\Rightarrow$ H.W

Example: 1) Evalute $\lim_{n\to\infty}\left(1+\frac{i}{n}\right)$

Sol: $z_n = 1+\frac{i}{n}$

$\Rightarrow x_n = 1$, $y_n = \frac{1}{n}$

$$\lim_{n\to\infty} x_n = 1 \quad , \lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} z_n = 1+0 = 1.$$

Definition

(1) A infinite series of complex number is an infinite sum

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots$$

(2) If the sequence $S_N = \sum_{n=1}^{N} z_n$, $N = 1, 2, \dots$ of partial sums converges to S, then we say that the series of complex numbers $\sum_{n=1}^{\infty} z_n$ converges to S, and we write

$$\sum_{n=1}^{\infty} z_n = S$$

(3) If a series of complex numbers does not converage then, we say that it diverges.

Theorem:- let $z_n = x_n + iy_n$ and $S = X + iY$. Then

$$\sum_{n=1}^{\infty} z_n = S = X + iY$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$$

proof:- H.W

Proposition:-

(1) If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$

(2) If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges

proof:- ① Assume $\sum_{n=1}^{\infty} z_n$ converges $\Rightarrow \sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$

Converage

$\Rightarrow \lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$

$\Rightarrow \lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n + iy_n) = 0$

proof:- ② Assume $\sum_{n=1}^{\infty} |z_n|$ converges

$|x_n| \le |z_n|$
$|y_n| \le |z_n|$ } By D.C.T

$\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converage

$\Rightarrow \sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge.

$\Rightarrow \sum_{n=1}^{\infty} z_n$ converges

Def. - If $\sum_{n=1}^{\infty} z_n = S$, then we define the remainder

$$r_N = S - S_N.$$

Remark: We note that a series converges to S if and only if

$$\lim_{N \to \infty} r_N = 0$$

if and only if

$$\lim_{N \to \infty} |r_N| = 0$$

Def. - A power series is a series of the form $\sum_{n=1}^{\infty} a_n (z - z_0)^n$, where z_0 and a_n are complex constants and z is a complex variable

Example 1) Show that if $|z| < 1$, then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

Sol. - $S_N = 1 + z + \cdots + z^{N-1}$

$$zS_N = z + z^2 + \cdots + z^N$$

$$zS_N - S_N = z^N - 1 \Rightarrow S_N = \frac{z^N - 1}{z - 1}$$

$$R_N = S - S_N$$

$$= \frac{1}{1-z} - \frac{1-z^N}{1-z} = \frac{z^N}{1-z}$$

$$|R_N| = \frac{|z|^N}{|1-z|}$$

$$\lim_{N\to\infty} |R_N| = \lim_{N\to\infty} \frac{|z|^N}{|1-z|}$$

$$= \lim_{N\to\infty} \frac{e^{N \ln |z|}}{|1-z|} = 0 \quad \text{, since } |z| < 1$$

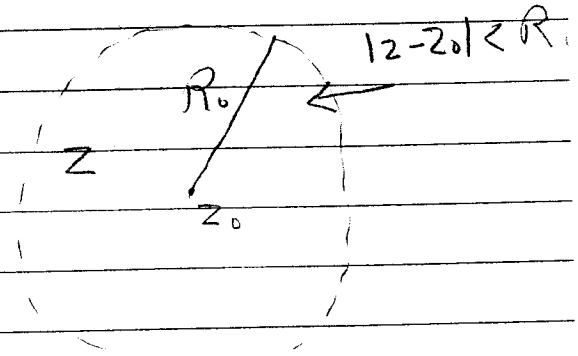
Taylor's theorem

Theorem. (Taylor's theorem):-

Suppose that a function f is analytic throughout an open disk $|z - z_0| < R_0$. Then, at each point z in that disk, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$. That is, the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, converges to $f(z)$ in the disk $|z - z_0| < R_0$.

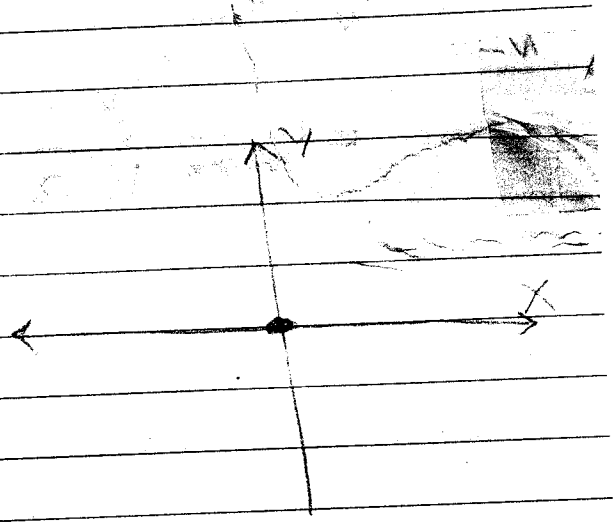


$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Remark:- According to Taylor's theorem, the Taylor series of a function f about z_0 converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 where f fails to be analytic.

Example:- 1) Show that the Maclaurin series expansion of $f(z) = e^z$ is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty.$$



Sol:- $f(z) = e^z$, $z_0 = 0$

e^z is an analytic function $\Rightarrow R = \infty$

we want to write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, z^n}{n!}$$

$f(z) = e^z \Rightarrow f(0) = 1$

$f'(z) = e^z \Rightarrow f'(0) = 1$

$\vdots$

$f^{(n)}(z) = e^z \Rightarrow f^{(n)}(0) = 1 \quad \forall n$

$$\Rightarrow f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad |z| < \infty \text{ or } z \in \mathbb{C}.$$

$$f(z) = \sum \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Example: 2) Find the Taylor series expansion of $f(z) = \cos z$ about $z_0 = 0$

Sol: $f(z) = \cos z, \quad z_0 = 0$

$f(z)$ is entire $\Longrightarrow$

$$\cos z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad |z| < \infty.$$

$f(z) = \cos z \Rightarrow f(0) = 1$

$f'(z) = -\sin z \Rightarrow f'(0) = 0$

$f''(z) = -\cos z \Rightarrow f''(0) = -1$

$f'''(z) = \sin z \Rightarrow f'''(0) = 0$

$n = 2m+1 \Longrightarrow \boxed{f^{(2m+1)}(0) = 0}, \quad m = 0, 1, 2, \ldots$

$n = 2m \Longrightarrow \boxed{f^{(2m)}(0) = (-1)^m}, \quad m = 0, 1, 2, \ldots$

$$\cos z = \sum_{m=0}^{\infty} \frac{f^{(2m+1)}(0)}{(2m+1)} z^{2m+1} + \sum_{m=0}^{\infty} \frac{f^{(2m)}(0)}{(2m)!} z^{2m}$$

$$\cos z = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} z^{2m}, \quad z \in \mathbb{C}.$$

Some Important Maclaurin Series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad |z| < \infty.$$

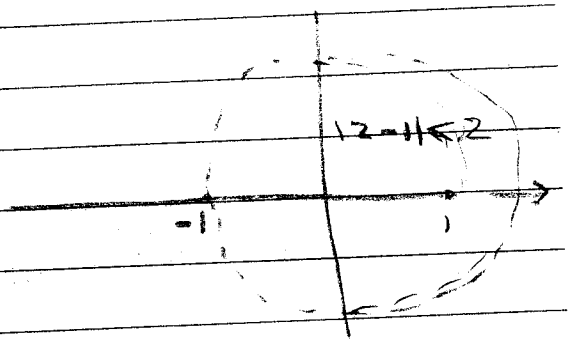
$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad , \forall z \in \mathbb{C}.$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad , \forall z \in \mathbb{C}.$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , |z| < 1.$$

Example 1):- Use $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z|<1$, to find the Taylor series expansion of

$$f(z) = \frac{1}{z+1}, \text{ about } z_0 = 1$$



Sol:- $f(z) = \frac{1}{z+1}$, $z_0 = 1$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (z-1)^n$$

$|z-1|<2$

$$f(z) = \frac{1}{z+1} = \frac{1}{(z-1+2)} = \frac{1}{2\left[\frac{z-1}{2}+1\right]} = \frac{1}{2\left[1-\left(\frac{1-z}{2}\right)\right]}$$

$$\Rightarrow \frac{1}{z+1} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1-z}{2}\right)^n \quad , \left|\frac{1-z}{2}\right| < 1$$

$$\frac{1}{z+1} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^n}{2^n} \quad , \frac{|z-1|}{2} < 1$$

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n \quad , |z-1| < 2$$

Example 2) Find the Taylor series representation of

$$f(z) = (z-\pi)\sin z.$$

Sol. $f(z) = (z-\pi)\sin z \quad , z_0 = \pi.$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(z-\pi)^n, \quad |z-\pi| < \infty$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad |z| < \infty.$$

$$\Rightarrow \sin(z-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}(z-\pi)^{2n+1}, \quad |z-\pi| < \infty$$

$$\sin(z-\pi) = \sin z \cos\pi - \sin\pi \cos z$$

$$= -\sin z - \sin\pi \cos z$$

$$\Rightarrow \sin z = -\sin(z-\pi) \quad (\sin\pi = 0)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-\pi)^{2n+1}$$

$$f(z) = (z-\pi)\sin z$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(z-\pi)^{2n+2}, \quad |z-\pi| < \infty.$$

Lurant series المقارنة بين Taylor series و

$$L.T. \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad , \quad T.S. \Rightarrow \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= a_0 + a_1 (z-z_0) + \dots$$

Laurent series

In many applications we encounter functions that are not analytic at a given point z_0. In that case we can not apply Taylor theorem. However, another series representation can be found in which both positive and negative powers of $z-z_0$ exist.

Def:- A Laurent series is a series of the form

$$\sum_{k=-\infty}^{\infty} c_k (z-z_0)^k = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Remark: $$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

if we let $n=-m$

$$\therefore \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{m=\infty}^{1} c_{-m} (z-z_0)^{-m} + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

$$= \sum_{m=1}^{\infty} \frac{c_{-m}}{(z-z_0)^m} + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Example 1): $\sum_{n=1}^{\infty} z^{n-2} = \frac{1}{z} + 1 + z + z^2 + \dots$ is

a Laurent series. (لانها احتوت على حدود سالبة الذي هو اكد الاول.)

Def:- A Laurent series

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

converges if both $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ and $\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$ converge.

Remark:- We can consider $\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$ as a power series for $w = (z-z_0)^{-1}$. If it converges for $|w| < R_1$, then $\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$ converges for $\frac{1}{|z-z_0|} < R_1$ or $|z-z_0| > \frac{1}{R_1}$

Therefore, if $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges for $|z-z_0| < R_2$, then the Laurent series converges for $\frac{1}{R_1} < |z-z_0| < R_2$

Example 1):- Consider the Laurent series

$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

The series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ converges $|z| < 2$ and the series $\sum_{n=1}^{\infty} z^{-n}$ converges when $|z^{-1}| < 1$ or $|z| > 1$.

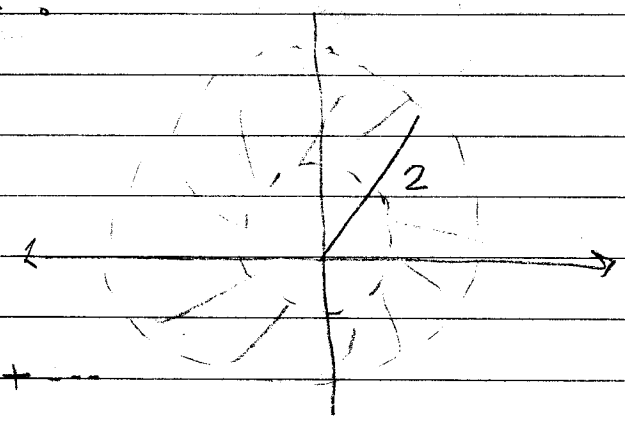
Therefore $\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ converges when

Sol: $\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=-\infty}^{\infty} c_n z^n$

$$c_n = \begin{cases} 1 & , n \le -1 \\ \frac{1}{2^{n+1}} & , n \ge 0 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \frac{1}{2} \frac{1}{1-\frac{z}{2}}$$

$$= \frac{1}{2-z}, \quad |z| < 2.$$



Now

$$\sum_{n=1}^{\infty} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \cdots\right] = \frac{1}{z} \frac{1}{1-\frac{1}{z}}, \quad \left|\frac{1}{z}\right| < 1$$

$$\sum_{n=1}^{\infty} z^n = \frac{1}{z-1}, \quad |z| > 1$$

Thus

$$\sum_{n=-\infty}^{\infty} c_n z^n = \frac{1}{2-z} + \frac{1}{z-1}$$

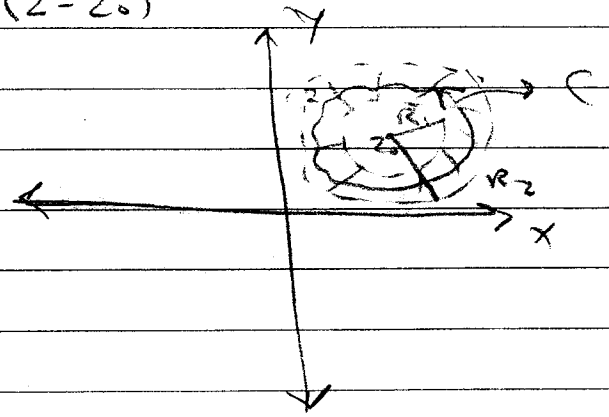
converges for $1 < |z| < 2$

Theorem (Laurent's theorem) Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ and let C be any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point z in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$



Remark: If f is analytic inside the disc $|z-z_0| < R_2$, then

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \text{ implies}$$

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad \text{for } n \geq 0$$

For $n < 0$, $f(z)(z-z_0)^{-(n+1)}$ is analytic within and on C and hence

$$c_n = 0, \quad \text{for } n \leq -1.$$

Therefore if f is analytic inside the disc $|z-z_0| < R_2$, then the Laurent series reduces to the Taylor series about z_0.

Example 1) Find the Laurent series expansion of

$$f(z) = \frac{\cosh z}{z^3} \quad \text{for } 0 < |z| < \infty.$$

Then use it to evaluate $\int_C \frac{\cosh z}{z^3}\,dz$, where C is the positively oriented circle $|z| = 1$.

Sol:- $f(z) = \frac{\cosh z}{z^3}$, $0 < |z| < \infty$ $(R_1 < |z - z_0| < R_2)$

$\Rightarrow z_0 = 0$

Thus $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, $R_1 < |z| < R_2$

$$\cosh z = \cos(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (iz)^{2n}, \quad |z| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} i^{2n} z^{2n}, \quad |z| < \infty$$

$$i^{2n} = (i^2)^n = (-1)^n$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!} z^{2n}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \quad |z| < \infty$$

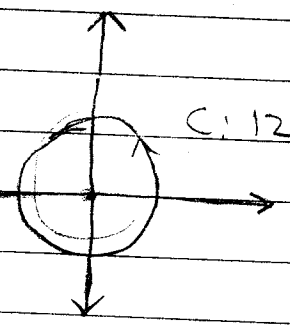
$$\Rightarrow f(z) = \frac{1}{z^3} \cosh z$$

$$= \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \quad 0 < |z| < \infty$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n-3}, \quad 0 < |z| < \infty$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz,$$

$n = -1$: $c_{-1} = \frac{1}{2\pi i} \int_C f(z) dz$



$$\Rightarrow \int_C f(z) dz = 2\pi i \, c_{-1} = \frac{2\pi i}{2} = \pi i$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n-3} = z^{-3} + \frac{1}{2!} z^{-1} + \frac{1}{4!} z + \cdots$$

c_{-1}

Example 2) Find the Laurent series representation of

$$f(z) = z e^{1/z}$$

in the domain, $0 < |z| < \infty$.

Sol:- $f(z) = z e^{1/z}$, $0 < |z| < \infty$

$\Rightarrow z_0 = 0$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty$$

$$\Rightarrow e^{1/z} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!}, \quad \left|\frac{1}{z}\right| < 0$$

$$\Rightarrow e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}, \quad |z| > 0$$

$$\Rightarrow \boxed{f(z) = z e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{1-n}, \quad 0 < |z| < \infty}$$

$$= z + 1 + \frac{1}{2} z^{-1} + \frac{1}{3!} z^{-2} + \cdots$$

Example:- Find the Laurent series that represents

$$f(z) = \frac{1}{z(z-1)}$$ in the domain, $0 < |z-1| < 1$.

Sol:- $f(z) = \frac{1}{z(z-1)}$, $0 < |z-1| < 1 \Rightarrow z_0 = 1$

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n (z-1)^n} \; ??$$

$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1-(1-z)}$$

w

Finite

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}, \quad |w| < 1$$

$$\Rightarrow \frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n, \quad |1-z| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad |z-1| < 1$$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad |z-1| < 1$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^{n-1} = (z-1)^{-1} - 1 + (z-1) + \cdots$$

Example:- Find the Laurent series that represents

$$f(z) = \frac{1}{z(z-1)} \text{ in the domain, } 0 < |z| < 1.$$

Sol:-

$$f(z) = \frac{1}{z(z-1)}, \quad 0 < |z| < 1$$

$$\Rightarrow z_0 = 0 \Rightarrow \boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n z^n} \; ??$$

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

$$\Rightarrow f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} \sum_{n=0}^{\infty} z^n$$

$$\Rightarrow \frac{1}{z(z-1)} = \boxed{-\sum_{n=0}^{\infty} z^{n-1}}, \quad 0 < |z| < 1$$

Example:- Find the Laurent series that represents

$$f(z) = \frac{1}{(z+1)(z-2)} \text{ in the domain, } 1 < |z| < 2.$$

Sol: $f(z) = \dfrac{1}{(z+i)(z-2)}$, $1 < |z| < 2$

$\Rightarrow z_0 = 0$, $\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n z^n}$??

$$\frac{1}{(z+i)(z-2)} = \frac{A}{z+i} + \frac{B}{z-2}$$

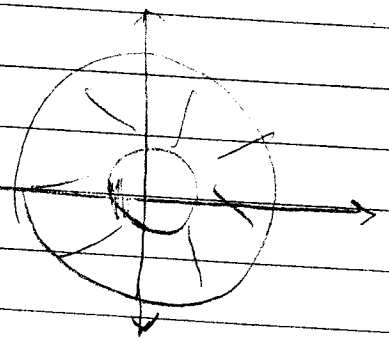
$$1 = A(z-2) + B(z+i)$$

$z = 2 \Rightarrow 1 = B(2+i) \Rightarrow B = \dfrac{1}{2+i} = \boxed{\dfrac{2-i}{5}}$

$z = -i \Rightarrow 1 = A(-i-2) \Rightarrow A = \dfrac{-1}{2+i} = -B$

$$\Rightarrow f(z) = \frac{-B}{z+i} + \frac{B}{z-2}$$

$$f(z) = \frac{2-1}{5}\left[\frac{1}{z-2} - \frac{1}{z+i}\right]$$



$$\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{-1}{2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \quad \left|\frac{z}{2}\right| < 1$$

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad |z| < 2$$

$$\frac{1}{z+i} = \frac{1}{z\left(1+\frac{i}{z}\right)} = \frac{1}{z\left[1-\left(\frac{-i}{z}\right)\right]} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-i}{z}\right)^n, \quad \left|\frac{-i}{z}\right| < 1$$

$$\frac{1}{z+i} = \sum_{n=0}^{\infty}\frac{(-i)^n}{z^{n+1}}, \quad |z| > 1$$

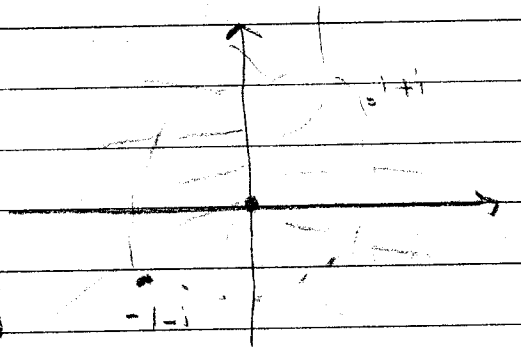
$1 < |z| < 2$

$$f(z) = \frac{2-i}{5}\left[\frac{1}{z-2} - \frac{1}{z+i}\right] \Rightarrow f(z) = \left[-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{(-i)^n}{z^{n+1}}\right]$$

Example: Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z(z^2-2i)}$$

and specify the regions in which those expansions are valid



Sol: $f(z) = \frac{1}{z(z^2-2i)} = \sum_{n=-\infty}^{\infty}$ on

$f(z)$ is analytic $\forall z \in \mathbb{C} \setminus \{0, \pm(-1+i)\}$

$0 < |z| < r_2$, $r_2 < |z| < \infty$

① in the region $0 < |z| < \sqrt{2}$

$$f(z) = \frac{1}{z(z^2-2i)}$$

$$\frac{1}{z^2-2i} = \frac{-1}{2i\left(1-\frac{z^2}{2i}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z^2}{2i}\right)^n, \left|\frac{z^2}{2i}\right| < 1$$

$$= \frac{i}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2i}\right)^n z^{2n}, \quad |z^2| < 2$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n i^{n-1} z^{2n}}{2^{n+1}}, \quad |z| < \sqrt{2}$$

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n i^{n+1} z^{2n-1}}{2^{n+1}}, \quad 0 < |z|$$

② In $\sqrt{2} < |z| < \infty$

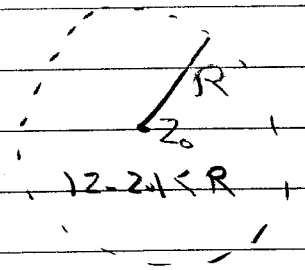
$$f(z) = \frac{1}{z(z^2-2i)} = \frac{1}{z^3\left(1-\frac{2i}{z^2}\right)} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{2i}{z^2}\right)^n, \left|\frac{2i}{z^2}\right| < 1$$

$|z^2| > 2$

$$= \sum_{n=0}^{\infty} \frac{(2i)^n}{z^{2n+3}} \quad , \quad |z| > \sqrt{2}$$

Integration and differentiation of power series.

Def. The greatest circle $|z - z_0| = R$ such that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at each point inside it is called the circle of convergence.



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Theorem: (Integration of power series)

Let C be any contour lying inside the circle of convergence of the power series.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = S(z).$$

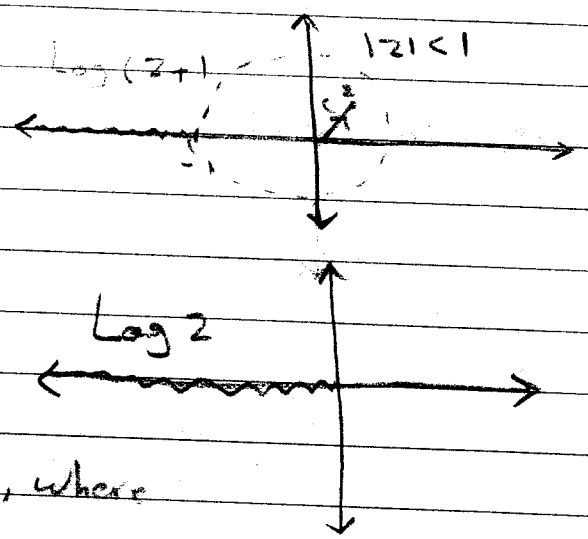
and let g be any function that is continuous on C. Then the series

$$g(z) S(z) = \sum_{n=0}^{\infty} a_n g(z) (z - z_0)^n$$

can be integrated term by term over C, that is

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz$$

Ex, Use $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$, $|z|<1$ to find the Maclaurin series expansion of $f(z) = \text{Log}(z+1)$.



Sol: $f(z) = \text{Log}|z+1|$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

Let C be the line segment from 0 to z, where $|z|<1$. Then

$$\int_C \frac{1}{1+w}\,dw = \sum_{n=0}^{\infty} (-1)^n \int_C w^n\,dw, \quad |z|<1$$

$$\text{Log}(1+w)\Big|_0^z = \sum_{n=0}^{\infty} (-1)^n \frac{w^{n+1}}{n+1}\Big|_0^z$$

$$\Rightarrow \text{Log}(1+z) - \text{Log}\,1 = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}$$

$$\Rightarrow \text{Log}(z+1) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} \quad |z|<1$$

Corollary: If $S(z) = \sum_{a_n}^{\infty} a_n (z-z_0)^n$ has a circle of convergence $|z-z_0| = R$, then $S(z)$ is analytic at each point in the open disc $|z-z_0| < R$.

Remark: This corollary is helpful in estabishing the analyticity of functions and evaluating limits.

Example: Let $f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$

(a) Show that f is an entire function

(b) Evalute $\lim_{z \to 0} f(z)$.

Sol: (a) $f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < \infty$.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$$

$$\Rightarrow \frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}, \quad 0 < |z| < \infty$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

$$f(0) = 1 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} \Big|_{z=0}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}, \text{ for all } z \in \mathbb{C}.$$

Thus $f(z)$ is an entire function

(b) $f(z)$ is entire $\Rightarrow f(z)$ is continuous $\forall z \in \mathbb{C}$

$$\Rightarrow \lim_{z \to 0} f(z) = f(0) = 1$$

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

Theorem (Differentiation of power series)

The powe series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ can be differentiated term by term. That is, at each z inside the circle of convergence of that series

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Example. Find the Maclaurin series expansion of $f(z)$

$$f(z) = \frac{1}{(z-1)^2}.$$

Sol: $f(z) = \frac{1}{(z-1)^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

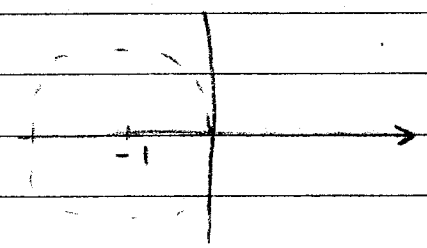
$$\Rightarrow \frac{d}{dz}\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} \frac{d}{dz} z^n, \quad |z| < 1$$

$$\Rightarrow \boxed{\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1}, \quad |z| < 1}$$

Example. Find the Taylor series expansion of $f(z)$

$$f(z) = \frac{1}{z^3} \text{ about } z = -1$$

Sol: $f(z) = \frac{1}{z^3} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (z+1)^n$



$$\frac{1}{z} = \frac{1}{z+1-1} = \frac{-1}{1-(z+1)}$$

$$\frac{1}{z} = -\sum_{n=0}^{\infty} (z+1)^n, \quad |z+1| < 1$$

$$\frac{-1}{z^2} = \sum_{n=1}^{\infty} n(z+1)^{n-1}$$

$$\frac{2}{z^3} = \sum_{n=2}^{\infty} n(n-1)(z+1)^{n-2}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{z^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(z+1)^{n-2}, \quad |z+1| < 1}$$

$m = n-2$

$$f(z) = \frac{1}{z^3} = \frac{1}{2}\sum_{n=0}^{\infty} (m+2)(m+1)(z+1)^m$$

Uniqueness of series representations

Theorem. (Uniqueness of Taylor series)

If a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ inside some open disc $|z-z_0| < R$, then it is the Taylor series expansion for f about z_0.

Theorem (Uniqueness of Laurent series):

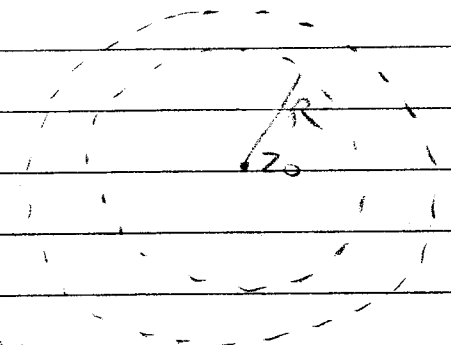
If a series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ converges to $f(z)$ inside some annular domain $R_1 < |z-z_0| < R_2$, then it is the Laurent series expansion for f in that domain.

Multiplication and division of power Series

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n = f(z)$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n = g(z)$ be convergent within some circle $|z-z_0| = R$ Then f and g

are analytic in the disk $|z-z_0|<R$.
Thus their product $f(z)g(z)$ is analytic in the disk $|z-z_0|<R$ and has a Taylor series expansion

$$f(z)g(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n, \quad |z-z_0|<R.$$



By definition:

$$c_0=f(z_0)g(z_0)=a_0b_0$$

$$c_1=\frac{d}{dz}\left[f(z)g(z)\right]_{z=z_0}=a_0b_1+a_1b_0,$$

$$c_2=\frac{1}{2!}\frac{d^2}{dz^2}\left[f(z)g(z)\right]_{z=z_0}=a_0b_2+a_1b_1+a_2b_0+\cdots$$

Using Leibniz rule, we have

$$c_n=\frac{1}{n!}\frac{d^n}{dz^n}\left[f(z)g(z)\right]_{z=z_0}$$

$$=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{d^k}{dz^k}f(z)\right]_{z=z_0}\left[\frac{d^{n-k}}{dz^{n-k}}g(z)\right]_{z=z_0}$$

$$=\sum_{k=0}^{n}\frac{f^{(k)}(z_0)g^{(n-k)}(z_0)}{k!\,(n-k)!}=\sum_{k=0}^{n}a_kb_{n-k}$$

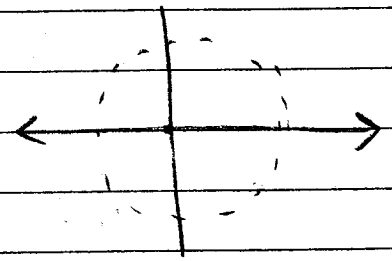
therefore

$$f(z)g(z)=\left[\sum_{n=0}^{\infty}a_n(z-z_0)^n\right]\left[\sum_{n=0}^{\infty}b_n(z-z_0)^n\right]$$

$$=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}a_kb_{n-k}\right](z-z_0)^n.$$

Example: Find the Maclaurin series expansion of

$$f(z) = \frac{e^z}{1-z}$$



Sol: $f(z) = \frac{e^z}{1-z} = \sum_{n=0}^{\infty} c_n z^n, \quad |z|<1$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z|<\infty$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$$

$$f(z) = \frac{e^z}{1-z} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} z^n\right), \quad |z|<1$$

$$= \left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right]\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$\Rightarrow \frac{e^z}{1-z} = 1 + \left[1+1\right]z + \left[1+1+\frac{1}{2}\right]z^2 + \left[1+1+\frac{1}{2}+\frac{1}{6}\right]z^3 + \cdots$$

$$= 1 + 2z + \frac{5}{2}z^2 + \frac{16}{6}z^3 + \cdots$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_n b_{n-k}\right) z^n$$

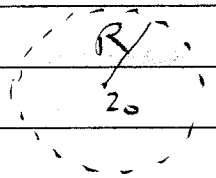
$$f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\right) z^n$$

Division of power series:

Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z)$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n = g(z)$

be convergent within some circle $|z-z_0| = R$. If $g(z) \neq 0$ for all z inside $|z-z_0| < R$, then $\frac{f(z)}{g(z)}$ is analytic throughout the disc $|z-z_0| < R$ and hence it has a Taylor series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n, \quad |z-z_0| < R.$$

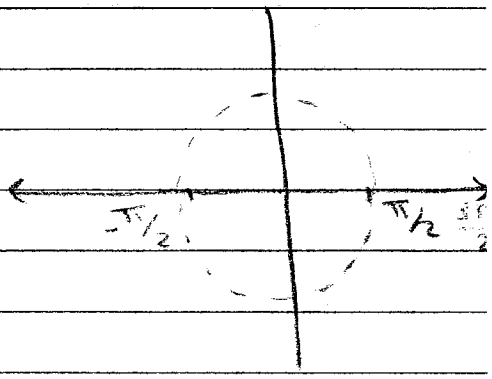


where $d_n = \frac{d^n}{dz^n}\left[\frac{f(z)}{g(z)}\right]$. The same result can be obtained by formally dividing the first series by the second.

Example:- Use division to show that

$$\sec z = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots \qquad |z| < \frac{\pi}{4}$$

Sol:- $f(z) = \sec z = \sum_{n=0}^{\infty} d_n z^n, \quad |z| < \frac{\pi}{2}$



$$f(z) = \sec z = \frac{1}{\cos z}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} d_n z^n$$

$$\Rightarrow 1 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} d_n z^n\right)$$

$$1 = \left[1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right]\left[d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots\right]$$

$$1 = d_0 + d_1 z + \left[d_2 - \frac{1}{2}d_0\right]z^2 + \left[d_3 - \frac{1}{2}d_1\right]z^3 + \left[d_4 - \frac{1}{2}d_2 + \frac{1}{4!}d_0\right]z^4 + \cdots$$

z^0: $1 = d_0$

z: $0 = d_1$

z^2: $0 = d_2 - \frac{1}{2}d_0 \Rightarrow d_2 = \frac{1}{2}$

z^3: $0 = d_3 - \frac{1}{2}d_1 \Rightarrow d_3 = 0$

z^4: $0 = d_4 - \frac{1}{2}d_2 + \frac{1}{24}d_0 \Rightarrow d_4 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$

Chapter 6
Residues and poles

The Cauchy-Goursat theorem states that the integral around a simple closed contour C of a function that is analytic within and on that contour C is zero. In this chapter, our aim is to study methods to evalute these integral when the function fails to be analytic at a finite number of points interior to C.

Residues:-

Recall that a point z_0 is called a singular point of a function f if f is not analytic at z_0 but is analytic at som point in every neighborhood of z_0.

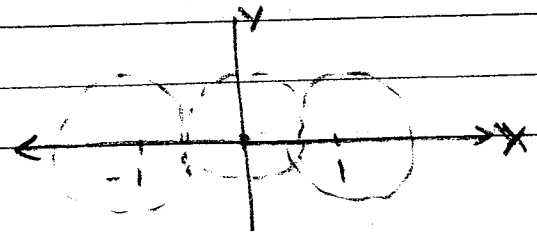
Def:- A singular point of a function f is called an isolated singular point if there is a deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0 of z_0 throughout which f is analytic.

Example: The function

$$f(z) = \frac{z-1}{z(z^2-1)}$$

has three singular points $z = 0, z = \mp 1$. These singular points are isolated

Sol:- $f(z) = \dfrac{z-1}{z(z^2-1)}$ is analytic $\forall z \in \mathbb{C} - \{0, 1, -1\}$

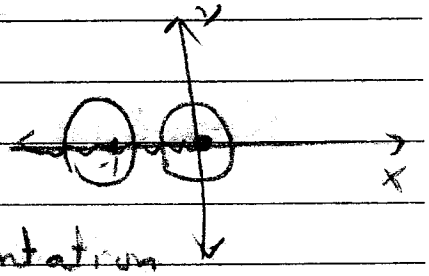


Ex:- The function

$$f(z) = \text{Log } z$$

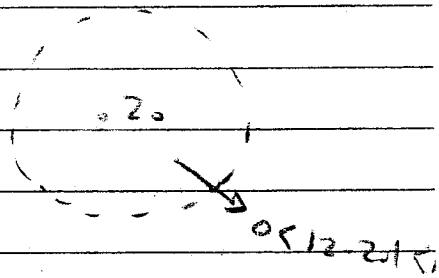
has a singular point at $z = 0$. However this singularity is not isolated since every neighborhood of $z = 0$ contains points on the negative real axis where f fails to be analytic.

sol:- $f(z) = \text{Log } z$ is analytic $\forall z \in \mathbb{C} - \{z : \text{Im}(z) = 0, \text{Re}(z) \leq 0\}$



Let f has isolated singular point at z_0. Then there is a number R such that f is analytic in the domain $0 < |z - z_0| < R$. Consequently f has a Laurent series representation in that domain

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$



where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and C is any positively oriented simple closed con

when $n = -1$, this expression for c_n be written

$$\int_C f(z)\, dz = 2\pi i\, c_{-1}.$$

Def:- Let z_0 be an isolated singular point of a function f and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

be the Laurent series representation of f in the domain

$0 < |z - z_0| < R$.

Then c_{-1} is called the residue of f at z_0 and it is denoted by

$$\text{Res}_{z=z_0} f(z).$$

Example: Let $f(z) = \frac{1}{z(z-2)}$. Find $\text{Res}_{z=0} f(z)$.

Sol: $f(z) = \frac{1}{z(z-2)}$ is analytic

$z = 2$, $z = 0$ are isolated singular point

$\text{Res}_{z=0} f(z) = c_{-1}$??

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 < |z| < R$$

$$f(z) = \frac{1}{z(z-2)}$$

$$\frac{1}{z-2} = \frac{-1}{2\left(1 - \frac{z}{2}\right)} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \qquad \left|\frac{z}{2}\right| < 1$$

$$f(z) = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad 0 < |z| < 2$$

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^{n-1}}{2^{n+1}}, \quad 0 < |z| < 2$$

$$= -\left[\frac{z^{-1}}{2} + \frac{1}{4} + \frac{z}{8} + \cdots\right]$$

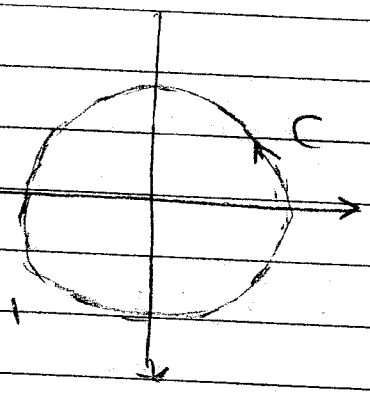
$$\text{Res}_{z=0} f(z) = c_{-1} = -\frac{1}{2}$$

Residue theorem

Theorem 1 (Cauchy's residue theorem)
Let C be a positively oriented simple closed contour. If a function f is analytic inside and on C except for a finite number of singular points z_k, $k = 1, 2, \dots, n$, inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=0}^{\infty} \operatorname*{Res}_{z=z_k} f(z)$$

Example. Evalute $\int_C z^2 e^{1/z}\,dz$, where C is the positively oriented circle $|z|=1$



Sol: $f(z) = z^2 e^{1/z}$, $C: |z| = 1$
$f(z)$ is analytic $\forall z \in \mathbb{C} - \{0\}$
In particular $f(z)$ is analytic on $0 < |z| \le 1$
The only singular point is $z_1 = 0$
By Residue theorem

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

$$\operatorname*{Res}_{z=0} f(z) = c_{-1} = ??$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 < |z| < \infty$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty$$

$$\Rightarrow e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}, \quad \left|\frac{1}{z}\right| < \infty$$

$$\Rightarrow f(z) = z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^{n-2} n!}, \quad 0 < |z| < \infty$$

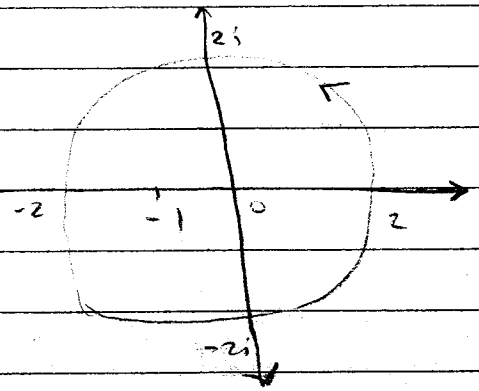
$$= \sum_{n=0}^{\infty} \frac{z^{2-n}}{n!}$$

$$= z^2 + z + \frac{1}{2} + \frac{z^{-1}}{3!} + \frac{z^{-2}}{4!} + \cdots$$

$$\Rightarrow \boxed{\operatorname*{Res}_{z=0} f(z) = \frac{1}{3!} = \frac{1}{6}}$$

$$\Rightarrow \int_C z^2 e^{1/z}\, dz = \frac{2\pi i}{6} = \frac{\pi i}{3}$$

Ex. Evaluate $\int_C \frac{z+2}{z(z+1)}\, dz$, where C is the positively oriented circle $|z|=2$.



Sol: $f(z) = \frac{z+2}{z(z+1)}$

$C: |z|=2$

$f(z)$ is analytic $\forall z \in \mathbb{C} - \{0, -1\}$

$z=0$, $z=-1$ are inside C

$$\Rightarrow \int_C f(z)\, dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z) \right]$$

$\operatorname*{Res}_{z=0} f(z) = ??$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 < |z| < 1$$

$$f(z) = \frac{z+2}{z(z+1)} = \frac{z+2}{z} = \frac{1}{1-(-z)}$$

$$f(z) = \frac{z+2}{z} \sum_{n=0}^{\infty} (-1)^n z^n, \quad 0 < |z| < 1$$

$$= \left(1 + \frac{2}{z}\right) \sum_{n=0}^{\infty} (-1)^n z^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n + 2 \sum_{n=0}^{\infty} (-1)^n z^{n-1} \quad 0 < |z| < 1$$

$$= [1 - z + z^2 + \dots] + 2[z^{-1} - 1 + z - z^2 + \dots]$$

$\operatorname{Res}_{z=0} f(z) = 2$

$\operatorname{Res}_{z=-1} f(z) = ??$, $f(z) = \sum_{n=-\infty}^{\infty} c_n (z+1)^n$, $0 < |z+1| < 1$

$$f(z) = \frac{z+2}{z(z+1)} = \frac{z+2}{z+1}\,\frac{1}{z+-1} = \left(1 + \frac{1}{z+1}\right)\frac{(-1)}{1-(z+1)}$$

$$= -\left(1 + \frac{1}{z+1}\right)\sum_{n=0}^{\infty} (z+1)^n \quad , \quad 0 < |z+1| < 1$$

$$f(z) = -\left[\sum_{n=0}^{\infty} (z+1)^n + \sum_{n=0}^{\infty} (z+1)^n\right]$$

$$= -\left[1 + (z+1) + (z+1)^2 + \dots\right]$$

$$\Rightarrow \operatorname{Res}_{z=-1} f(z) = -1$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\,[z-1] = 2\pi i$$

Theorem 2 (Residue at infinity)

Let C be a positively oriented simple closed contour. If a function f is analytic everywhere in the finite plane except for a finite number of singular points inside C, then

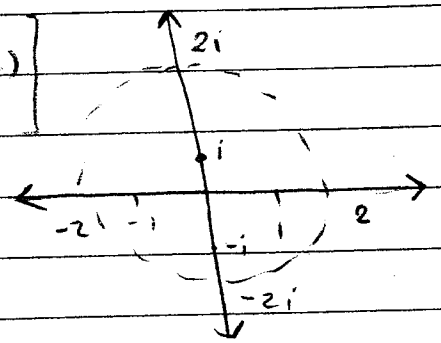
$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Ex:- Evalute $\int_C \frac{1}{(z-1)(z^2+1)}\,dz$, where C is the positively oriented circle $|z|=2$.

Sol:- $f(z)=\frac{1}{(z-1)(z^2+1)}$ is analytic $\forall z \in \mathbb{C} - \{1, \mp i\}$

The singular point $1, \mp i$ are inside $C: |z|=2$.

$$\int_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right]$$



$$= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{1}{\left(\frac{1}{z}-1\right)\left(\frac{1}{z^2}+1\right)}$$

$$= \frac{1}{\left(\frac{1-z}{z}\right)(1+z^2)}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{-z}{(z-1)(z^2+1)}$$

$\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = c_{-1}$??

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0<|z|<1$$

$\frac{1}{z^2} f\left(\frac{1}{z}\right)$ is analytic at $z=0$

$\Rightarrow \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 0$

Thus $\int_C f(z)\,dz =$

The three types of isolated singular points

In this section, we will classify the isolated singular point $z=z_0$ of a function f according to the part

$$\sum_{n=-\infty}^{-1} c_n (z-z_0)^n$$

of the Laurent series representation of f in the domain

$$0 < |z-z_0| < R$$

{T.S $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$, L.S $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$}

Def: Let z_0 be an isolated singular point of a function f and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

be the Laurent series representation of f in the domain $0 < |z-z_0| < R$. Then the portion

$$\sum_{n=-\infty}^{-1} c_n (z-z_0)^n$$

involving negative powers of $z-z_0$, is called the principle part of f at z_0.

Example: Find the principle part of f at $z=1$ if

$$f(z) = \frac{1}{(z-1)^2(z+1)}$$

Sol: P.P. of f at $z=1$

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-1)^n \quad \text{in } 0 < |z-1| < 2$$

$$\frac{1}{z+1} = \frac{1}{(z-1+2)} = \frac{1}{2\left(1+\frac{(z-1)}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$$

$$\left|\frac{z-1}{2}\right| < 1$$

$$\frac{1}{z+1} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{2^{n+1}}, \quad |z-1| < 2$$

$$\Rightarrow f(z) = \frac{1}{(z-1)^2(z+1)}$$

$$= \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{2^{n+1}}, \quad 0 < |z-1| < 2$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^{n-2}}{2^{n+1}}, \quad 0 < |z-1| < 2$$

$$= \frac{(z-1)^{-2}}{2} - \frac{(z-1)^{-1}}{2^2} + \frac{1}{2^3} + \cdots$$

$\Rightarrow$ The principle part of $f(z)$ at $z=1$ is

$$\frac{1}{2}(z-1)^{-2} - \frac{1}{4}(z-1)^{-1}$$

Def:- Let z_0 be an isolated singular point of a function f and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

be the Laurent series representation of f in the domain $0 < |z-z_0| < R$.

1- If $c_n = 0$ for all $n \leq -1$, then z_0 is called a removable singular point.

2- If there exists a negative integer m such that $c_m \neq 0$ and $c_n = 0$, for $n < m$, then z_0 is called a pole of order $-m$.

3- If an infinite number of the coefficients $c_n (n < 0)$ are nonzero, then z_0 is called an essential singular point.

Example: Find the isolated singular points of

$$f(z) = \frac{\cos z}{z^5}$$

and determine whether they are poles removable, or essential singular points.

sol. $f(z) = \frac{\cos z}{z^5}$ is analytic $\forall z \in \mathbb{C} - \{0\}$

The only isolated singular point is $z = 0$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 < |z| < \infty$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad |z| < \infty$$

$$\Rightarrow f(z) = \frac{\cos z}{z^5} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-5}}{(2n)!} \quad 0 < |z| < \infty$$

$$f(z) = z^{-5} - \frac{1}{2!} z^{-3} + \frac{1}{4!} z^{-1} - \frac{1}{6!} z + \cdots$$

$\Rightarrow z = 0$ is a pole of order 5.

Ex: Find and classify the isolated singular points of

$$f(z) = \frac{\sin z - 1}{2z - \pi}.$$

Sol: $f(z) = \frac{\sin z - 1}{2z - \pi}$ is analytic $\forall z \in \mathbb{C} - \{\frac{\pi}{2}\}$

$z_1 = \frac{\pi}{2}$ is an isolated singular point

To classify $z = \frac{\pi}{2}$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - \frac{\pi}{2})^n, \quad 0 < |z - \frac{\pi}{2}| < \infty$$

$$\sin z = \sin(z - \frac{\pi}{2} + \frac{\pi}{2}) = \sin(z - \frac{\pi}{2})\cos\frac{\pi}{2} + \cos(z - \frac{\pi}{2})$$

$$\Rightarrow \sin z = \cos(z - \frac{\pi}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z - \frac{\pi}{2})^{2n}, \quad |z - \frac{\pi}{2}| < \infty$$

$$\Rightarrow \sin z - 1 = \left[1 - \frac{1}{2!}(z - \frac{\pi}{2})^2 + \dots\right] - 1$$

$$\sin z - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (z - \frac{\pi}{2})^{2n}$$

$$f(z) = \frac{\sin z - 1}{2z - \pi} = \frac{1}{2(z - \frac{\pi}{2})} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (z - \frac{\pi}{2})^{2n}$$

$$f(z) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (z - \frac{\pi}{2})^{2n-1}, \quad 0 < |z - \frac{\pi}{2}| < \infty$$

$$f(z) = \frac{1}{2}\left[\frac{-1}{2!}(z - \frac{\pi}{2}) + \frac{1}{4!}(z - \frac{\pi}{2})^3 + \dots\right]$$

$\Rightarrow z = \frac{\pi}{2}$ is a removable singular point.

Example:- Find and classify the singular points of $f(z) = \sin\frac{1}{z}$.

Sol:- $f(z) = \sin\left(\frac{1}{z}\right)$ is analytic $\forall z \in \mathbb{C} - \{0\}$

$z = 0$ is the only isolated singular point

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 < |z| < \infty$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$$

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{z}\right)^{2n+1}}{(2n+1)!}, \quad \left|\frac{1}{z}\right| < \infty$$

$$\Rightarrow f(z) = \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-n-1}}{(2n+1)!}, \quad |z| > 0$$

$$= z^{-1} - \frac{1}{3!} z^{-3} + \frac{1}{5!} z^{-5} + \cdots$$

$\Rightarrow z = 0$ is an essential singular point.

Remark:-

1. A pole of order 1 is called a simple pole.

2. If a function f has a removable singular point at z_0, then the function $F(z) = \begin{cases} f(z), & z \neq z_0 \\ c_0, & z = z_0 \end{cases}$ is analytic in some neighborhood of z_0.

3. In any neighborhood of an essential singular point, a function assume every finite value, with one

passipale exceptron, an infinite number of times.

Example: Show that $f(z) = \frac{\cos z - 1}{z}$ has a removable singular point at $z = 0$ and redefine $f(z)$ so that it is analytic at $z = 0$.

Sol: $f(z) = \frac{\cos z - 1}{z}$, $\text{Domain}(f) = \mathbb{C} - \{0\}$

$z = 0$ is an isolated singular point

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} z^{2n}$$

$$\Rightarrow \cos z - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \quad |z| < \infty.$$

$$f(z) = \frac{\cos z - 1}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-1} \quad 0 < |z| < \infty$$

$$f(z) = \frac{-1}{2!} z + \frac{1}{4!} z^3 - \cdots$$

$\Rightarrow z = 0$ is removable

let $F(z) = \begin{cases} f(z), & z \neq 0 \\ 0, & z = 0 \end{cases}$

$$F(z) = -\frac{1}{2!} z + \frac{1}{4!} z^3 + \cdots, \quad 0 < |z| < \infty$$

$F(0) = 0$